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# *Surfaces with the Same Spherical Representation of their Lines of Curvature as Pseudospherical Surfaces.*

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## INTRODUCTION.

Surfaces whose Gaussian curvature is negative and constant—pseudospherical surfaces—have been the object of study of the geometers since the days of Gauss, and as the result of their study there exists to-day an extensively developed theory of the properties of these surfaces and the transformations by which groups of them can be obtained from one of them. The character of the lines of curvature on these surfaces is of interest, and especially the curves on the unit sphere which serve for their spherical representation. It can be shown that when these lines are parametric, the linear element of the sphere is reducible to the form

$$d\sigma^2 = \sin^2 \omega \, du^2 + \cos^2 \omega \, dv^2, \tag{a}$$

where $\omega$ is a particular solution of

$$\frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega . \tag{b}$$

In spite of the large amount of work which has been done upon these surfaces very little attention has been directed to the surfaces other than pseudospherical whose lines of curvature have this representation upon the sphere. It is our purpose to make a study of these surfaces and we shall find that many of the the theorems for pseudospherical surfaces have significant analogues in this broader field. For convenience, we shall refer to them as the *A-surfaces.* After the manner of Codazzi, we write the linear element of the surface in the form

$$ds^2 = A^2du^2 + C^2dv^2. \tag{c}$$

In §1, several methods are discussed by means of which one can determine from a given $A$-surface all the surfaces with the same representation of their lines of curvature. It is found that the simplest solution of this problem reduces to the integration of a partial differential equation of the Laplace type. It will be convenient to refer to all the surfaces with the same spherical representation of their lines of curvature as forming a *group* or *class.*

In §2, we consider particular $A$-surfaces and notice first of all that the *moulure* surfaces are of this kind. When the functions $A$ and $C$ in (c) are given the respective values $\cos\omega$ and $\sin\omega$, the corresponding surface is pseudospherical. In seeking to determine all the $A$-surfaces for which $A$ and $C$ are functions of $\omega$ alone, we find that they are the pseudospherical surfaces and their parallels. The section closes with the determination of the $A$-surfaces, which are Weingarten surfaces.

The method of the *transformation of Lie* for pseudospherical surfaces can be applied at once to all the $A$-surfaces, with the result that when one of these surfaces is known, the spherical representation of an infinity of groups of new $A$-surfaces can be found directly; this is considered in §3.

Bianchi was the first to call attention to the fact that, if upon a surface of total curvature minus one, a family of parallel geodesics be given and upon tangents to these curves points be taken at unit distance from the point of contact, the locus of these points is a surface of the same total curvature and the lines are tangent to the second surface as well as to the first. The lines of curvature upon the new surface correspond to the similar lines on the original surface, and the linear element of the spherical representation of the former is given by (a) when $\omega$ is replaced by $\theta$, the latter function being the angle between the tangent to the geodesic and the line of curvature $v = \text{const.}$ on the first surface. The determination of this angle $\theta$ requires the integration of a Riccati equation, so that there is an infinity of such families of geodesics. Darboux has given the name *transformations of Bianchi* to the foregoing. But one of these new surfaces can be defined also as the envelope of the plane perpendicular to the tangent plane to the surface, through the point of contact and meeting the tangent plane in the line which makes an angle $\theta$ with the tangent to the line of curvature, $v$=const. It is found that the Riccati equation for the determination of this function $\theta$ involves only $\omega$ and hence has a meaning for the $A$-surfaces. If such a family of curves is taken upon any $A$-surface $S$ and the corresponding envelope of the plane, drawn as in the transformation of pseudospherical surfaces, is determined, it is found that it is an $A$-surface whose lines of curvature are in correspondence with the lines of curvature upon $S$ and have the spherical representation whose linear element is given by (a) when $\omega$ is replaced by $\theta$. Since $\theta$ involves an arbitrary constant, there is an infinity of transforms of each $A$-surface. In turn, each of these surfaces can be transformed in an infinity of ways, and we shall see later that the functions giving the transformation are obtained by algebraic operations and quadratures.

The development of the preceding theory is contained in §4, and in §5 we consider some of its results. It is found that only in the case of pseudospherical surfaces does the point of tangency of the envelope fall on the intersection of the two planes, so that the curves determined by the angle $\theta$ are geodesics only in this case. We then investigate the problem of finding when the lines joining corresponding points on a surface and its transform constitute a normal congruence. We have seen that through each point $M$ of an $A$-surface $S$ an infinity of planes can be drawn which envelop the transforms of $S$. We find that all the points of contact are on a circle whose axis is the normal to $S$ at $M$, which gives an interesting interpretation of the cyclic systems of circles associated with every $A$-surface.

Bianchi has considered the group of surfaces which have the following property: *The sphere described on every segment of the normal comprised between the two centers of curvature as diameters, cut a fixed sphere in great circles, or orthogonally, or pass through the center.*[*] To these three types he has given the respective names, *elliptic*, *hyperbolic* and *parabolic*. We shall refer to them as the *surfaces of Bianchi.* After having derived in §6 the expressions for the coordinates of these surfaces in a manner similar to that followed by Bianchi and noted that they are $A$-surfaces, we determine their transforms. These are shown to be surfaces of Bianchi of the parabolic type.

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[*] Nuove ricerche sulle superficie pseudosferiche, Annali di Matematica, Ser. 2, Vol. 24, p. 347.

In §7, we discover generalized transformations of $A$-surfaces suggested by the generalization by Bäcklund of the transformations of Bianchi for pseudospherical surfaces. The transformations of Bäcklund may be defined as follows: Upon a pseudospherical surface $S$ there is an infinity of families of curves which are such that, if planes be drawn making a certain constant angle $\sigma$ with the tangent planes to $S$, passing through the points of contact and intersecting the latter planes in the tangent lines to the curves of one of these families, these planes envelop a surface of the same total curvature as $S$. These curves may be determined by the angle $\theta$ which their tangents make with the tangents to the lines of curvature $v = \text{const.}$, and the complete determination of $\theta$ requires the solution of a Riccati equation whose coefficients involve $\sigma$ and $\omega$, where the latter is the function entering in the spherical representation (a) of $S$. From this it follows that $\theta$ can be used in connection with all $A$-surfaces having the same representation. If, for all of these surfaces, we construct planes as in the case of the pseudospherical surface of the same group, their envelopes are $A$-surfaces and all of them form a group. In this manner, after the integration of a Riccati equation, we can get, from one surface $S$, a doubly infinite system of $A$-surfaces, for $\sigma$ and the constant of integration are arbitrary. Each of these surfaces belongs to a different group of surfaces, and the complete integration of a partial differential equation of the second order puts us in a position for finding all the surfaces of all these groups by direct operations.

Bianchi has shown,[*] that if a pseudospherical surface $S$ be transformed into $S_1$ and $S_2$ so that the angles between the tangent planes to $S$ and to the latter shall be respectively $\sigma_1$ and $\sigma_2$, there is known a function $\phi$ by means of which $S_1$ and $S_2$ can be transformed into the same surface $S_3$ so that the angles between the tangent planes to $S_3$ and to $S_1$ and $S_2$ respectively are $\sigma_2$ and $\sigma_1$. We show that this theorem is true for all $A$-surfaces, in consequence of which we have that when one knows how to transform an $A$-surface in a general manner, the transformations of the transforms can be effected by algebraic processes and differentiation.

General transformations of surfaces of Bianchi are considered in §9, and it is found that surfaces of the parabolic type and these alone are transformed into surfaces of this kind whatever be $\sigma$.

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[*] Lezioni, p. 435; Ger. trans., p. 461.

§10 is devoted to the study of surfaces corresponding to the solution $\omega = 0$ of equation (b) and their transforms. It closes with an investigation of the transforms of the moulure surfaces.

## I.

### §1.—*A-Surfaces. Definition and Determination.*

Consider a surface $S$ upon which the lines of curvature are parametric and refer it to the trihedron whose $x$-, $y$-, $z$-axes are respectively the tangents to the curves $v$=const., $u$=const. at the point and the normal to the surface; the positive directions along these lines are so chosen that as one looks upon the tangent plane from a point on the positive part of the $z$-axis, the positive $y$-axis is to the left of the positive $x$-axis. If we write the linear elements of the surface and its spherical representation in the forms

$$ds^2 = A^2 du^2 + C^2 dv^2 \tag{1}$$

and

$$d\sigma^2 = q^2 du^2 + p_1^2 dv^2, \tag{2}$$

and write further

$$\rho_{gu} = A/r, \quad \rho_{gv} = C/r_1, \tag{3}$$

where $\rho_{gu}$ and $\rho_{gv}$ denote the geodesic curvature of the curves $v =$ const. and $u =$ const. respectively, one can find the following necessary and sufficient relations between these several functions:*

$$\frac{\partial p_1}{\partial u} = -q r_1, \quad \frac{\partial q}{\partial v} = r p_1, \quad \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -q p_1,$$

$$r = -\frac{1}{C}\frac{\partial A}{\partial v}, \quad r_1 = \frac{1}{A}\frac{\partial C}{\partial u}.$$

We consider the surfaces for which

$$d\sigma^2 = \sin^2\omega\, du^2 + \cos^2\omega\, dv^2. \tag{4}$$

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* Darboux, Lecons, Vol. 2, pp. 386, 392.

If we substitute these values for $p_1$ and $q$ in the above formulæ, we have the following fundamental relations:

$$p_1 = \cos\omega\,, \quad q = \sin\omega\,, \quad r = \frac{\partial\omega}{\partial v}\,, \quad r_1 = \frac{\partial\omega}{\partial u}\,, \tag{5}$$

$$-\frac{1}{C}\,\frac{\partial A}{\partial v} = \frac{\partial\omega}{\partial v}\,, \quad \frac{1}{A}\,\frac{\partial C}{\partial v} = \frac{\partial\omega}{\partial u} \tag{6}$$

and

$$\frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2} = \sin\omega\cos\omega\,. \tag{7}$$

From the preceding, it follows that every solution of this equation leads to as many surfaces of this kind as there are solutions $A$ and $C$ of the corresponding equations (6). From (4) it follows that all the surfaces arising from the same function $\omega$ have the same spherical representation of their lines of curvature; hence, the tangent planes at corresponding points of all these surfaces are parallel and the tangents to homologous lines of curvature at these points are parallel. Moreover, from the manner in which we defined these surfaces, it follows that every surface which has its lines of curvature homologous to the similar lines on an *A-surface* and at the same time corresponds with parallelism of tangent planes, is an $A$-surface.

Suppose that we have given two $A$-surfaces $S_1$ and $S_2$, referred to their lines of curvature. Denoting by $W_1$ and $W_2$ the distances from the origin upon the tangent planes to $S_1$ and $S_2$ at corresponding points, we have that $W_1$ and $W_2$ are particular solutions of the equation[*]

$$\frac{\partial^2\psi}{\partial u\,\partial v} - \frac{\partial\log\sin\omega}{\partial v}\,\frac{\partial\psi}{\partial u} - \frac{\partial\log\cos\omega}{\partial u}\,\frac{\partial\psi}{\partial v} = 0\,, \tag{8}$$

and the point-coordinates of these surfaces are given by replacing $\psi$ by $W_1$ and $W_2$ respectively, in the formulæ[*]

$$x = \psi X + \Delta(\psi, X), \quad y = \psi Y + \Delta(\psi, Y), \quad z = \psi Z + \Delta(\psi, Z), \tag{9}$$

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[*] Bianchi, Lezioni, p. 137 ; Ger. trans., p. 140.

where $X$, $Y$, $Z$ denote the direction-cosines of the normal to the surface with respect to fixed axes, and $\Delta(\psi, \theta)$ is the mixed differential parameter formed with respect to the quadratic form (4). Since the tangent planes to these surfaces are parallel, it follows that the difference between $W_2$ and $W_1$, denoted by $\lambda$, is a solution of equation (8). If, then, we have an $A$-surface with rectangular coordinates, $x$, $y$, $z$, referred to fixed axes and we can find a solution of (8), we have directly a second $A$-surface defined by

$$\xi = x + X\lambda + \Delta(\lambda, X), \quad \eta = y + Y\lambda + \Delta(\lambda, Y), \quad \zeta = z + Z\lambda + \Delta(\lambda, Z), \tag{10}$$

for $X$, $Y$, $Z$ are known in this case.

In particular, equation (8) is satisfied by a constant. Then (10) defines a parallel surface, so that we have the evident theorem: *All the parallel surfaces of an A-surface are A-surfaces.*

Since the tangents to homologous lines of curvature at corresponding points on two $A$-surfaces are parallel, we have

$$\frac{\partial x_2}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \quad \frac{\partial x_2}{\partial v} = \mu \frac{\partial x_1}{\partial v}, \tag{11}$$

and similar relations in $y$ and $z$. From these we get

$$A_2 = \lambda A_1, \quad C_2 = \mu C_1.$$

From (6) it follows that

$$\frac{1}{C_1}\frac{\partial A_1}{\partial v} = \frac{1}{C_2}\frac{\partial A_2}{\partial v}, \quad \frac{1}{A_1}\frac{\partial C_1}{\partial u} = \frac{1}{A_2}\frac{\partial C_2}{\partial u},$$

which, by means of the above, can be replaced by

$$\left.\begin{aligned} &\frac{\partial \lambda}{\partial v} + (\lambda - \mu)\frac{\partial \log A_1}{\partial v} = 0, \\ &\frac{\partial \mu}{\partial u} + (\mu - \lambda)\frac{\partial \log C_1}{\partial u} = 0. \end{aligned}\right\} \tag{12}$$

Hence, when an $A$-surface, $S_1$, is given, the complete determination of all surfaces with the same spherical representation of their lines of curvature requires the solution of the system (12) and quadratures (11).

The system (12) can be replaced by a slightly different one in which the coefficients involve $\omega$ directly. Thus, in consequence of the Rodrigues formulæ,[*]

$$\frac{\partial x_1}{\partial u} = \rho_1 \frac{\partial X}{\partial u}, \quad \frac{\partial x_1}{\partial v} = \rho_2 \frac{\partial X}{\partial v},$$

where $\rho_1$ and $\rho_2$ denote the principal radii of curvature of $S_1$, we have

$$A_2 = \frac{\rho_1'}{\rho_1} A_1, \quad C_2 = \frac{\rho_2'}{\rho_2} C_1,$$

where $\rho_1'$, $\rho_2'$ are the radii of curvature for $S_2$. But[†]

$$\rho_1 = -\frac{A}{q}, \quad \rho_2 = \frac{C}{p_1}, \tag{13}$$

so that the above reduce to

$$A_2 = -l \sin \omega, \quad C_2 = m \cos \omega,$$

where, for the moment, we have put $l$ and $m$ for $\rho_1'$ and $\rho_2'$. If these values be substituted in the equations for $S_2$ of the form (6), one has for the determination of $l$ and $m$,

$$\left.\begin{aligned} &\frac{\partial l}{\partial v} + (l - m) \frac{\partial \log \sin \omega}{\partial v} = 0, \\ &\frac{\partial m}{\partial u} + (m - l) \frac{\partial \log \cos \omega}{\partial u} = 0. \end{aligned}\right\} \tag{14}$$

When $l$ and $m$ are equal, they are constants, in which case $S_2$ is a sphere. If we make an exception of this evident solution and eliminate $m$ from these equations and again $l$, we get

$$\left.\begin{aligned} &\frac{\partial^2 l}{\partial u \partial v} - \frac{\partial \log \sin \omega}{\partial v} \frac{\partial l}{\partial u} - \left[\frac{\partial}{\partial u} \log \cos \omega + \frac{\partial}{\partial u} \log \left(\frac{\partial \log \sin \omega}{\partial v}\right)\right] \frac{\partial l}{\partial v} = 0, \\ &\frac{\partial^2 m}{\partial u \partial v} - \left[\frac{\partial}{\partial v} \log \sin \omega + \frac{\partial}{\partial v} \log \left(\frac{\partial \log \cos \omega}{\partial u}\right)\right] \frac{\partial m}{\partial u} \\ &\qquad\qquad - \frac{\partial}{\partial u} \log \cos \omega \frac{\partial m}{\partial v} = 0. \end{aligned}\right\} \tag{15}$$

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[*] Bianchi, Lezioni, p. 101; Ger. trans., p. 102.

[†] Darboux, Leçons, Vol. 2, p. 392.

When a solution of either of these equations is obtained, the other function corresponding is found from (14) by two quadratures.

All of these processes for the determination of all surfaces with the same representation of their lines of curvature as a given $A$-surface come back to the solution of a partial differential equation of the second order. But the subsequent steps in the first introduce the least difficulty, and, consequently, we are inclined to give preference to it.

### §2.—*Particular A-surfaces.*

The necessary and sufficient condition that the two equations (15) be the same, in which case they have the form (8), is that the two equations

$$\frac{\partial^2}{\partial u\,\partial v}\log\sin\omega = 0, \quad \frac{\partial^2}{\partial u\,\partial v}\log\cos\omega = 0,$$

be satisfied. From these it follows that

$$\sin^2\omega = U_1 V_1, \quad \cos^2\omega = U_2 V_2,$$

where $U_1$, $U_2$ are functions of $u$ alone and $V_1$, $V_2$ functions of $v$ alone. These functions must satisfy the condition

$$U_1 V_1 + U_2 V_2 = 1, \tag{16}$$

which, by differentiation with respect to $u$, becomes

$$U_1' V_1 + U_2' V_2 = 0,$$

from which it follows that either $U_1$ and $U_2$ are constants, or

$$U_1 = cU_2 + d, \quad V_2 = -cV_1,$$

where $c$ and $d$ are constants. When these values are substituted in equation (16), one finds that $V_1$ and $V_2$ are constants. Hence, the above conditions are satisfied only in case $\omega$ is a function of a single variable, say $u$. Then, from (14), one has that $l$ is a function of $u$ alone and $m$ is given by two quadratures, so that all these surfaces are given by quadratures. Let us consider them more minutely.

Since $\omega$ is a function of $u$ alone, we have from (6) that $A$ is also a function of $u$ alone, and, consequently, the lines of curvature $v = \text{const.}$ are geodesics.[*] They, therefore, belong to the class of surfaces, considered by Monge,[†] whose lines of curvature in one system are plane and the surface is generated by this curve when its plane envelops a developable surface. Again, the coefficients in (4) are functions of $u$ alone, so that the lines of curvature are represented on the sphere by a system of great circles with a common diameter. Hence, the generating developable is cylindrical, so that the surfaces are so-called *moulure* surfaces.

Now, equation (7) reduces to

$$\frac{d^2\omega}{du^2} = \sin\omega\cos\omega,$$

from which, by integration, we have

$$\left(\frac{d\omega}{du}\right)^2 = \sin^2\omega + a, \tag{17}$$

where $a$ is an arbitrary constant. In general, the integral of (17) is an elliptic function of $u$, but when $a$ is zero, the integral is

$$\tan\frac{\omega}{2} = ce^u. \tag{18}$$

Now equation (8) can be integrated; the complete integral is

$$\psi = \cos\omega\, U + V,$$

where $U$ and $V$ are arbitrary functions of $u$ and $v$ respectively. For this case, $X$, $Y$, $Z$ are given by quadratures, and after these have been determined, all the corresponding moulure surfaces are given directly by formulæ (9). The pseudo-spherical surfaces with this representation of their lines of curvature are surfaces of revolution, for their linear element is of the form (21). We shall return to a consideration of these surfaces.

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[*] Bianchi, Lezioni, p. 144; Ger. trans., p. 148.

[†] Application de l'Analyse à la Geométrie, 5th ed., p. 322.

Let us inquire now in what cases $A$ and $C$, as given by (6), are functions of $\omega$ alone. Put

$$A = \phi(\omega), \quad C = \psi(\omega),$$

then equations (6) become

$$\phi' + \psi = 0, \quad \psi' - \phi = 0.$$

Eliminating $\psi$, we get

$$\phi'' + \phi = 0,$$

so that

$$\left.\begin{aligned} A &= c_1 \cos\omega + c_2 \sin\omega, \\ C &= c_1 \sin\omega - c_2 \cos\omega, \end{aligned}\right\} \tag{19}$$

where $c_1$ and $c_2$ are arbitrary constants. If, in particular, we establish between these constants a relation such that we can put

$$c_1 = \cos\alpha, \quad c_2 = \sin\alpha,$$

where $\alpha$ is a constant (and thus consider a sub-class of the surfaces with the values (19)), we have

$$A = \cos(\omega - \alpha), \quad C = \sin(\omega - \alpha).$$

Hence, among all the surfaces of Bonnet arising from a solution $\omega$ of equation (7), there is a family whose linear element has the form

$$ds^2 = \cos^2(\omega - \alpha)\, du^2 + \sin^2(\omega - \alpha)\, dv^2, \tag{20}$$

where $\alpha$ is the parameter of the family. It is important to note that all the surfaces with this linear element belong uniquely to the class with the function $\omega$, for, when $\omega$ is a solution of equation (7), $\omega - \alpha$ cannot be an integral for any value of $\alpha$ except zero. When $\alpha$ is zero, the above linear element becomes

$$ds^2 = \cos^2\omega\, du^2 + \sin^2\omega\, dv^2. \tag{21}$$

But this is the well-known characteristic form of the linear element of a pseudospherical surface of curvature minus one referred to its lines of curvature.

It is readily seen that all the parallels of pseudospherical surfaces belong to the class for which $A$ and $C$ have the forms (19). For, if the constant distance be denoted by $a$, the following relations obtain among the fundamental coefficients $E, F, G$; $L, M, N$ of a surface and the similar functions $E_1, F_1, G_1$; $L_1, M_1, N_1$ of its parallel:*

$$\left.\begin{aligned} E_1 &= E\left(1-\frac{a^2}{\rho_1\rho_2}\right)+aL\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right], \\ F_1 &= F\left(1-\frac{a^2}{\rho_1\rho_2}\right)+aM\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right], \\ G_1 &= G\left(1-\frac{a^2}{\rho_1\rho_2}\right)+aN\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right]. \end{aligned}\right\} \tag{22}$$

For the pseudospherical surface referred to its lines of curvature, these quantities have the forms

$$\begin{aligned} E &= A^2 = \cos^2\omega, & F &= 0, & G &= C^2 = \sin^2\omega, \\ L &= \frac{A^2}{\rho_1} = \sin\omega\cos\omega, & M &= 0, & N &= \frac{C^2}{\rho_2} = -\sin\omega\cos\omega. \end{aligned}$$

From these we get by a ready calculation

$$E_1 = A_1^2 = (\cos\omega - a\sin\omega)^2, \qquad G_1 = C_1^2 = (\sin\omega + a\cos\omega)^2, \tag{23}$$

which are evidently of the form (19). These expressions for $A_1$ and $C_1$ differ from those of (19) by the same constant factor, so that we have the theorem:

*The only A-surfaces for which A and C are functions of $\omega$ alone are pseudospherical surfaces, their parallels and the homothetics of these.*

In order to find the $A$-surfaces whose principal radii are functions of one another, it is only necessary to recall the theorem of Weingarten* that this problem reduces to the determination of orthogonal systems on the sphere such that the corresponding linear element can be given the form

$$d\sigma^2 = \frac{du^2}{x^2} + \frac{dv^2}{\phi'^2(x)}, \tag{24}$$

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* Knoblauch, Einleitung in die Allgemeine Theorie der Krummen Flächen, p. 235.

where $\varkappa$ is a function of $u$ and $v$, and that when such a system is found, the principal radii of the surface with this representation of its lines of curvature are given by

$$\rho_1 = \phi(\varkappa), \quad \rho_2 = \phi(\varkappa) - \varkappa\phi'(\varkappa), \tag{25}$$

where the prime denotes differentiation. Comparing (24) with (4), one finds

$$\rho_1 = -\tan\omega + c, \quad \rho_2 = \cot\alpha + c, \tag{26}$$

where $c$ is a constant of integration. When $c$ is zero, the total curvature of the surface is minus one, and for other values of $c$ the surfaces are the parallels of the former. Hence the theorem:

*The pseudospherical surfaces and their parallels are the only Weingarten surfaces which are A-surfaces.*

## §3.—*Lie Transformations of A-Surfaces.*

Thus far we have considered only the lines of curvature as parametric. If, now, we effect the change of parameters given by

$$\alpha = \frac{u+v}{2}, \quad \beta = \frac{u-v}{2}, \tag{27}$$

the equation (7) takes the form

$$\frac{\partial^2\omega}{\partial\alpha\,\partial\beta} = \sin\omega\cos\omega. \tag{28}$$

Lie[†] has remarked that if $\omega(\alpha, \beta)$ is a solution of this equation, so also is $\omega\left(m\alpha, \frac{\beta}{m}\right)$ for any value, except zero, of the constant $m$. If, then, $\omega(u, v)$

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[*] Darboux, Leçons, Vol. 3, p. 319.

[†] Ueber Flächen deren Krümmungsradien durch eine Relation verknüpft sind, Archiv for Mathematik og. Naturvidenskab, Vol. 4, p. 510; also Darboux, l. c., p. 381.

is a solution of equation (7) and, consequently, $\omega(\alpha+\beta,\ \alpha-\beta)$ of equation (28), it follows that $\omega\left(\alpha m+\frac{\beta}{m},\ \alpha m-\frac{\beta}{m}\right)$ is a solution of the latter equation and, consequently,

$$\omega\left[\frac{(m^2+1)u+(m^2-1)v}{2m},\ \frac{(m^2-1)u+(m^2+1)v}{2m}\right]$$

is a solution of (7). If we put $\frac{1-\cos\sigma}{\sin\sigma}$ in place of $m$, this takes the much simpler form

$$\omega\left(\frac{u-v\cos\sigma}{\sin\sigma},\ \frac{v-u\cos\sigma}{\sin\sigma}\right), \tag{29}$$

where evidently $\sigma$ is a constant taking any values but 0 and $\pi$.

When this observation of Lie is considered in connection with the pseudospherical surfaces arising from the integration of equation (7), it is seen that the knowledge of one surface of this kind leads to an infinity of them. But this so-called *transformation of Lie* is equally applicable to any surface of Bonnet and leads to a more general result. Thus, if we have given any surface of Bonnet referred to its lines of curvature, and the linear element of the sphere in the form (4), we have the spherical representation of an infinity of groups of these surfaces, and the determination of them all requires the integration of equation (8), in which $\omega$ has been given the general form (29).

It is evident that when the given surface is a parallel of a pseudospherical surface, that is $A=\phi(\omega)$ and $C=\psi(\omega)$, the linear element of a Lie transform is obtained when $u$ and $v$ in $A$ and $C$ are replaced by $\frac{u-v\cos\sigma}{\sin\sigma}$ and $\frac{v-u\cos\sigma}{\sin\sigma}$ respectively. We inquire for all cases of this kind. If the above expressions be denoted by $u_1$ and $v_1$, we must have from (6)

$$-\frac{1}{C(u_1,v_1)}\ \frac{\partial A(u_1,v_1)}{\partial v}=\frac{\partial\omega(u_1,v_1)}{\partial v},\quad \frac{1}{A(u_1,v_1)}\ \frac{\partial C(u_1,v_1)}{\partial u}=\frac{\partial\omega(u_1,v_1)}{\partial u}, \tag{30}$$

of which the first reduces to

$$\frac{1}{C(u_1,v_1)}\left[\frac{\partial A(u_1,v_1)}{\partial u}\cot\sigma-\frac{1}{\sin\sigma}\ \frac{\partial A(u_1,v_1)}{\partial v_1}\right]$$
$$=-\frac{\partial\omega(u_1,v_1)}{\partial u_1}\cot\sigma+\frac{\partial\omega(u_1,v_1)}{\partial v_1}\ \frac{1}{\sin\sigma}. \tag{31}$$

But

$$-\frac{1}{C(u_1, v_1)}\frac{\partial A(u_1, v_1)}{\partial v_1} = \frac{\partial \omega(u_1, v_1)}{\partial v_1}, \tag{32}$$

so that the above reduces to

$$-\frac{1}{C(u_1, v_1)}\frac{\partial A(u_1, v_1)}{\partial u_1} = \frac{\partial \omega(u_1, v_1)}{\partial u_1}. \tag{33}$$

These two relations show that $A$ is a function of $\omega$; similarly, $C$ is a function of $\omega$. Hence, only for pseudospherical surfaces and their parallels does the above change of parameters give the linear element of the new surface.

If in the transformation (29) we replace $\sigma$ by $\pi - \sigma$, it becomes

$$\omega\left(\frac{u + v\cos\sigma}{\sin\sigma}, \frac{v + u\cos\sigma}{\sin\sigma}\right). \tag{34}$$

When the transformations indicated by (29) and (34) are applied consecutively to a function $\omega(u, v)$, one gets the latter again. Hence, if the Lie transformation of angle $\sigma$ be denoted by $L_\sigma$, this result may be indicated by

$$L_\sigma^{-1} = L_{\pi-\sigma}. \tag{35}$$

## §4.—*Complementary Transformations of A-Surfaces.*

We are now going to establish for $A$-surfaces transformations analogous to the *transformations of Bianchi* of pseudospherical surfaces. We begin with a short development of the latter.*

Consider a pseudospherical surface $S$ referred to the moving trihedron previously defined and whose fundamental functions and their relations are given by (21), (4), (5), (6) and (7). Through the point $M$ and in the tangent plane we draw a line making an angle $\theta$ with the $x$-axis. Denote by $M_1$ the point on this line at unit distance from $M$. Its rectangular coordinates with respect to the trihedron are evidently

$$\cos\theta, \quad \sin\theta, \quad 0.$$

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* Darboux, Leçons, Vol. 3, p. 426.

When $M$ moves over $S$, the projections of the displacement of $M_1$ on these axes are*

$$\left.\begin{aligned}&-\sin\theta d\theta+\cos\omega du-\left(\frac{\partial\omega}{\partial v}du+\frac{\partial\omega}{\partial u}dv\right)\sin\theta,\\&\cos\theta d\theta+\sin\omega dv+\left(\frac{\partial\omega}{\partial v}du+\frac{\partial\omega}{\partial u}dv\right)\cos\theta,\\&\cos\omega\sin\theta dv-\sin\omega\cos\theta du.\end{aligned}\right\}\tag{36}$$

We consider now the locus of the point $M_1$ which we denote by $S_1$, and we demand the conditions which $\theta$ must satisfy in order that the line $MM_1$ be tangent to $S_1$ at $M_1$ and the tangent plane to $S_1$ be perpendicular to the tangent plane to $S$. Then the direction-cosines of the tangent plane to $S_1$ at $M_1$ are

$$+\sin\theta,\quad -\cos\theta,\quad 0,\tag{37}$$

and since the tangent to the above displacement must be in this plane, we must have

$$d\theta+\left(\frac{\partial\omega}{\partial v}-\sin\theta\cos\omega\right)du+\left(\frac{\partial\omega}{\partial u}+\sin\omega\cos\theta\right)dv=0.$$

Equating to zero the coefficients of $du$ and $dv$, we have, for the determination of $\theta$, the two equations

$$\left.\begin{aligned}\frac{\partial\theta}{\partial u}+\frac{\partial\omega}{\partial v}&=\cos\omega\sin\theta,\\\frac{\partial\theta}{\partial v}+\frac{\partial\omega}{\partial u}&=-\sin\omega\cos\theta.\end{aligned}\right\}\tag{38}$$

If $\theta$ be eliminated from these equations, we are brought to equation (7), so that these equations are compatible. Moreover, if $\omega$ be eliminated, it is found that $\theta$ satisfies equation (7).

By means of (38) the above expressions for the projections of the displacement of $M_1$ can be put in the form

$$\left\{\begin{aligned}&\cos\theta(\cos\omega\cos\theta du+\sin\omega\sin\theta dv),\\&\sin\theta(\cos\omega\cos\theta du+\sin\omega\sin\theta dv),\\&\qquad\cos\omega\sin\theta dv-\sin\omega\cos\theta du,\end{aligned}\right.$$

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* Ib., Vol. 2, p. 385.

from which it follows that the linear element of $S_1$ is

$$ds_1^2 = \cos^2\theta du^2 + \sin^2\theta dv^2.$$

Since $\theta$ is a solution of equation (7), it follows that $S_1$ is a pseudospherical surface. Hence, if we draw through a point of a pseudospherical surface of unit curvature a line in the tangent plane and making an angle $\theta$, given by (38), with the tangent to the line of curvature $v =$ const., the locus of the point at unit distance on this line is a pseudospherical surface whose tangent plane is perpendicular to the corresponding tangent plane to $S$. Darboux calls this the *transformation of Bianchi.* Since the system (38) can be replaced by a Riccati equation,[*] it follows that $\theta$ involves an arbitrary constant, and, consequently, there is an infinity of such transforms.

The surface $S_1$, which is the transform of $S$ corresponding to a given angle $\theta$, may be defined, in another manner, as the envelope of the plane through $M$ perpendicular to the tangent plane to $M$ and meeting the latter in a line which makes an angle $\theta$ with the tangent to the line of curvature, $v =$ const. We shall consider the character of this envelope when $S$ is any $A$-surface. The coordinates of the point $M_1$, where this plane touches the envelope, are evidently of the form

$$\lambda\cos\theta, \quad \lambda\sin\theta, \quad \mu,$$

where $\lambda$ and $\mu$ are functions of $u$ and $v$, which have to be determined.

The projections on the moving axes of a displacement of $M_1$ have the following expressions:[†]

$$\left.\begin{aligned}
d\xi &= \cos\theta d\lambda - \lambda\sin\theta d\theta + A du + \mu\sin\omega du - \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\lambda\sin\theta,\\
d\eta &= \sin\theta d\lambda + \lambda\cos\theta d\theta + C dv + \left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)\lambda\cos\theta - \mu\cos\omega dv,\\
d\zeta &= d\mu + \lambda\cos\omega dv - \lambda\sin\omega\cos\theta du.
\end{aligned}\right\} \quad (40')$$

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[*] Bianchi, Lezioni, p. 429 ; Germ. trans., p. 454.

[†] Darboux, Leçons, Vol. 2, p. 385.

Since $\theta$ is to be a solution of the system (38), these expressions can be put in the form

$$\left.\begin{aligned}
d\xi &= \left(\cos\theta\,\frac{\partial\lambda}{\partial u} + A + \mu\sin\omega - \lambda\sin^2\theta\cos\omega\right)du \\
&\qquad + \left(\cos\theta\,\frac{\partial\lambda}{\partial v} + \lambda\sin\omega\sin\theta\cos\theta\right)dv, \\
d\eta &= \left(\sin\theta\,\frac{\partial\lambda}{\partial u} + \lambda\cos\theta\sin\theta\cos\omega\right)du \\
&\qquad + \left(\sin\theta\,\frac{\partial\lambda}{\partial v} + C - \mu\cos\omega - \lambda\cos^2\theta\sin\omega\right)dv, \\
d\zeta &= \left(\frac{\partial\mu}{\partial u} - \lambda\sin\omega\cos\theta\right)du + \left(\frac{\partial\mu}{\partial v} + \lambda\cos\omega\sin\theta\right)dv.
\end{aligned}\right\}\tag{40}$$

From the expressions (37) for the direction-cosines of the tangent plane to $S_1$, it follows that the above values must satisfy the equation

$$\sin\theta d\xi - \cos\theta d\eta = 0,$$

which by (40) reduces to

$$\sin\theta\,(\lambda\cos\omega - \mu\sin\omega - A)\,du + \cos\theta\,(C - \lambda\sin\omega - \mu\cos\omega)\,dv = 0.$$

Equating to zero the coefficients of $du$ and $dv$, we have

$$\left.\begin{aligned}
A &= \lambda\cos\omega - \mu\sin\omega, \\
C &= \lambda\sin\omega + \mu\cos\omega,
\end{aligned}\right\}\tag{41}$$

from which it follows that

$$\left.\begin{aligned}
\lambda &= \phantom{-}A\cos\omega + C\sin\omega, \\
\mu &= -A\sin\omega + C\cos\omega.
\end{aligned}\right\}\tag{42}$$

Substituting these values for $\lambda$ and $\mu$ in (40), the latter become

$$\left.\begin{aligned}
d\xi &= \cos\theta\left[\cos\omega\,\frac{\partial A}{\partial u} + \sin\omega\,\frac{\partial C}{\partial u} + \mu\,\frac{\partial\omega}{\partial u} + \lambda\cos\theta\cos\omega\right]du \\
&\qquad + \cos\theta\left[\cos\omega\,\frac{\partial A}{\partial v} + \sin\omega\,\frac{\partial C}{\partial v} + \mu\,\frac{\partial\omega}{\partial v} + \lambda\sin\theta\sin\omega\right]dv, \\
d\eta &= \sin\theta\left[\cos\omega\,\frac{\partial A}{\partial u} + \sin\omega\,\frac{\partial C}{\partial u} + \mu\,\frac{\partial\omega}{\partial u} + \lambda\cos\theta\cos\omega\right]du \\
&\qquad + \sin\theta\left[\cos\omega\,\frac{\partial A}{\partial v} + \sin\omega\,\frac{\partial C}{\partial v} + \mu\,\frac{\partial\omega}{\partial v} + \lambda\sin\theta\sin\omega\right]dv, \\
d\zeta &= \left[-\sin\omega\,\frac{\partial A}{\partial u} + \cos\omega\,\frac{\partial C}{\partial u} - \lambda\,\frac{\partial\omega}{\partial u} - \lambda\cos\theta\sin\omega\right]du \\
&\qquad + \left[-\sin\omega\,\frac{\partial A}{\partial v} + \cos\omega\,\frac{\partial C}{\partial v} - \lambda\frac{\partial\omega}{\partial v} + \lambda\cos\omega\sin\theta\right]dv.
\end{aligned}\right\}\tag{43}$$

From this we have for the linear element of $S_1$,

$$ds_1^2 = \left(\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} + \lambda \cos\theta\right)^2 du^2 + \left(\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} + \lambda \sin\theta\right)^2 dv^2. \quad (44)$$

We are now in a position to show that the surface $S_1$ is an $A$-surface. It has been seen that all the surfaces of Bonnet corresponding to the same function $\omega$ have parallel tangent planes and their lines of curvature in correspondence. Let $\Sigma$ denote the pseudospherical surface related in this manner to our original surface $S$, and $m$ denote the point on the former corresponding to $M$ on the latter. Since the trihedrons at $M$ and $m$ are parallel, the plane through $m$ perpendicular to the tangent plane at $m$ and forming an angle $\theta$ with the $x$-axis, envelopes the pseudospherical surface $\Sigma_1$, which is the Bianchi transform of $\Sigma$ for the angle $\theta$. Hence $S_1$ and $\Sigma_1$ correspond with parallelism of tangent planes. Again, the parametric curves on $S_1$ correspond to the lines of curvature on $S$, consequently to the similar lines on $\Sigma$ and, therefore, to the lines of curvature on $\Sigma_1$. Hence, the parametric lines on $S_1$, which form an orthogonal system (44), are represented on the sphere by such a system, so that these lines are lines of curvature for $S_1$. Thus $S_1$ corresponds with $\Sigma_1$ with parallelism of tangent planes and lines of curvature, and, therefore, is an $A$-surface.

If we write (44) in the form

$$ds_1^2 = A_1^2\,du^2 + C_1^2\,dv^2,$$

it is readily shown by (6) and (38) that

$$-\frac{1}{C_1}\frac{\partial A_1}{\partial v} = \frac{\partial \theta}{\partial v} \qquad \frac{1}{A_1}\frac{\partial C_1}{\partial u} = \frac{\partial \theta}{\partial u},$$

which verifies the above geometrical reasoning. Hence:

*Given an A-surface corresponding to a solution $\omega$ of equation* (7). *If planes be drawn perpendicular to the tangent planes through the point of contact and making an angle $\theta$, given by* (38), *with the tangents to the curves $v = const.$, they envelop a new A-surface corresponding to the solution $\theta$ of equation* (7).

In order to find all the transform of $S$ we must have the general integral $\theta$ of the system (38). If $\theta_1$ denote any particular integral, the general integral

$\theta$ is given by[*]

$$\cot\left(\frac{\theta-\theta_1}{2}\right)=\beta_1 e^{-\alpha_1}, \tag{45}$$

where $\alpha_1$ and $\beta_1$ are given by the quadratures

$$\left.\begin{aligned} d\alpha_1 &= \cos\theta_1 \cos\omega\, du + \sin\theta_1 \sin\omega\, dv, \\ d\beta_1 &= e^{\alpha_1}(\sin\theta_1\cos\omega\, du - \cos\theta_1 \sin\omega\, dv)\,; \end{aligned}\right\} \tag{46}$$

the right-hand members are readily found to be exact differentials when it is noted that $\theta_1$ satisfies (38). Since these quadratures are general, it follows that $\theta$ involves an arbitrary constant so that there is an infinity of transformations of $S$. But each of these transforms is only one of a group of $A$-surfaces with the same spherical representation of the lines of curvature. From (8) it follows that the complete determination of all these surfaces arising from a single $A$-surface requires the integration of the equation

$$\frac{\partial^2\psi}{\partial u\,\partial v}-\frac{\partial\log\sin\theta}{\partial v}\frac{\partial\psi}{\partial u}-\frac{\partial\log\cos\theta}{\partial u}\frac{\partial\psi}{\partial v}=0. \tag{47}$$

It is evident that $\theta$, as given by (45), is the same for all $A$-surfaces belonging to the same function $\omega$. Hence, when all the surfaces arising from the general integral of (47) are determined, we have among them the transforms of all the surfaces with the same spherical representation as $S$.

Proceeding as in the case of $S$, we have that the transformation of the transforms of $S$ and of all $A$-surfaces belonging to the same $\omega$ requires the solution of the system

$$\left.\begin{aligned} \frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial v} &= \cos\theta\sin\phi, \\ \frac{\partial\phi}{\partial v}+\frac{\partial\theta}{\partial u} &= -\sin\theta\cos\phi. \end{aligned}\right\} \tag{48}$$

Now $\phi_1=\omega+\pi$ is a solution of these equations, hence the general integral for surfaces corresponding to the angle $\theta_1$ is given by[†]

$$\cot\left(\frac{\phi-\phi_1}{2}\right)=-\gamma_1 e^{\alpha_1}, \tag{49}$$

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[*] Darboux, Leçons, Vol. 3, p. 458.

[†] Darboux, l. c.

where $\alpha_1$ is given by (46) and $\gamma_1$ by the quadrature

$$d\gamma_1 = e^{-\alpha_1}(\cos\theta_1 \sin\omega\, du - \sin\theta_1 \cos\omega\, dv). \tag{50}$$

Thus, by a quadrature, we have the general function $\phi$ for the transformation of surfaces which are transforms by $\theta_1$ of $S$ and the surfaces of its class.

The formula (45) can also be written

$$\cot\left(\frac{\theta-\theta_1}{2}\right) = -\beta e^{-\alpha},$$

where $\alpha$ and $\beta$ are given by (46) when $\theta_1$ is replaced by $\theta$. Between these functions exist the relations

$$e^{\alpha} = \frac{e^{\alpha_1}}{e^{2\alpha_1}+\beta_1^2}, \quad \beta = \frac{-\beta_1}{e^{2\alpha_1}+\beta_1^2},$$

so that $\alpha$ and $\beta$ are found by algebraic processes. Again, formula (49) can be written

$$\cot\left(\frac{\phi-\phi_1}{2}\right) = \gamma e^{\alpha}, \tag{49$'$}$$

where $\gamma$ is given by the quadrature

$$d\gamma = e^{-\alpha}(\cot\theta \sin\omega\, du - \sin\theta \cos\omega\, dv).$$

Hence, it requires this further quadrature to obtain the function $\phi$ giving the transformations of all the transforms of $S$ and the surfaces with the same representation of the lines of curvature. And all the other surfaces of this third general group of surfaces follow from the integration of an equation of the form (8) in which $\omega$ has been replaced by the general solution of (49$'$), which involves at least two arbitrary constants. We shall consider further transformations later.

## §5.—*Particular Transformations. Associated Cyclic Systems.*

We shall consider now the expressions for $\lambda$ and $\mu$. For $\lambda$ to be zero, we must have

$$A = l\sin\omega, \quad C = -l\cos\omega,$$

where $l$ is an auxiliary function. From (6) we find that $l$ is a constant, and, consequently, from (4) and (13), the corresponding surface is a sphere.

When $S$ is pseudospherical, $\lambda$ is unity and $\mu$ is zero. For the latter to be satisfied, we must have

$$A = l \cos \omega, \qquad C = l \sin \omega,$$

where $l$ is an auxiliary function. From (6) we have that $l$ is a constant; hence, only in the case of pseudospherical surfaces is $\mu$ zero. For this case, $S$ and $S_1$ are the focal nappes of a rectilinear congruence, which is normal, since the tangent planes are perpendicular. For none of the other $A$-surfaces do the lines of intersection of the tangent planes to $S$ and $S_1$ form a normal congruence. Hence:

*Upon the pseudospherical surfaces the curves* $\theta = \text{const.}$ *are geodesics, and on no other A-surfaces.*

We seek now all the cases for which $\lambda$ is constant. If we substitute the values of $A$ and $C$ as given by (41) in (6) and consider $\lambda$ constant, it is found that $\mu$ also is constant, unless $\omega$ is zero; similar results follow from the hypothesis that $\mu$ is constant. Now, the formulæ (41) take the form (19), so that *the only A-surfaces for which* $\lambda$ *and* $\mu$ *can be constant are pseudospherical surfaces and their parallels, excepting in the case where* $\omega$ *is zero.*[*]

This was an evident solution of the problem, for, from the manner in which parallel surfaces are associated with one another, it follows that a transform of angle $\theta$ for one is the same for all. Hence, if $a$ denotes the distance between the tangent planes to a pseudospherical surface and a parallel, for the latter $\lambda$ is the same as for the former and $\mu$ is plus or minus $a$ according to the direction.

It has been seen that the transforms of the parallels of a pseudospherical surface are surfaces of this kind. We seek the general conditions which a surface $S$ must satisfy in order the transforms are pseudospherical. From (39) and (44) it follows that we must have

$$\frac{\partial A}{\partial u} + C \frac{\partial \omega}{\partial u} + (\lambda - \varepsilon) \cos \theta = 0,$$
$$\frac{\partial C}{\partial v} - A \frac{\partial \omega}{\partial v} + (\lambda - \varepsilon) \sin \theta = 0,$$

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[*] This will be discussed later.

where $\varepsilon$ is plus or minus one. By means of (6) this can be written

$$\frac{\partial}{\partial u}(A^2+C^2)=2A(\varepsilon-\lambda)\cos\theta,\qquad \frac{\partial}{\partial v}(A^2+C^2)=2C(\varepsilon-\lambda)\sin\theta. \quad (51)$$

It is readily found that these equations are compatible when $\theta$ is a solution of equation (38). Then, from these equations and (42) it is seen that a necessary condition is that given by

$$\frac{1}{A^2}\left[\frac{\partial}{\partial u}(A^2+C^2)\right]^2+\frac{1}{C^2}\left[\frac{\partial}{\partial v}(A^2+C^2)\right]^2=4(A\cos\omega+C\sin\omega-\varepsilon)^2.$$

When this condition is satisfied and $\theta$ given by either of equations (51) satisfies equations (38), not only is the corresponding surface capable of such a transform but the transforming angle is given.

Equations (51) are satisfied identically when $A^2+C^2$ is constant and $\lambda$ is plus or minus one. For the former case we have

$$A=t\cos\sigma,\qquad C=t\sin\sigma,$$

where $t$ is constant and $\sigma$ is an auxiliary angle. If these values be substituted in (6), it is found that

$$\sigma=\omega-\alpha,$$

where $\alpha$ is a constant. Hence for $\lambda$ to be a constant and $A^2+C^2$ to be constant, the surface $S$ in both cases must be parallels of a pseudospherical surface. It is to be remarked that this is the only case where $\theta$ is not determined by equations (51). Hence, although there may be $A$-surfaces capable of a single pseudospherical transform, the pseudospherical surfaces and their parallels are the only surfaces which have an infinity of such transforms.

It has been noticed that the system (48), which gives the angle of transformation of the surfaces $S_1$, admits the solution $\phi_1=\omega+\pi$. The corresponding transform of $S_1$ has for its spherical representation the linear element (4) and, therefore, belongs to the group of the original surface $S$. When the latter is pseudospherical, it coincides with this new surface $S'$. Moreover, this is true only for this case, for when $\mu$ is not zero, the tangent planes to $S$ and $S'$ are not coincident but parallel. Therefore, if $S$ is an $A$-surface other than a pseudospherical surface, we can get another of the same class as soon as we have a transform $S_1$ of $S$. With this same function $\theta$ we can continue to get $A$-surfaces belonging to the original group unless two of these surfaces $S$ have the same transform, or we are brought to a pseudospherical surface or one of the surfaces reduces to a curve or point. Later, we shall have an example of this last exception.

As a verification of the fact the plane through $M_1$ and parallel to the tangent plane to $S$ at $M_1$ envelops an $A$-surface of the same class as $S$, we have only to show that the value for $\mu$ given by (42) is a solution of equation (8). This is readily done in consequence of the relations (6).

We shall seek now for the coordinates, with respect to the trihedron for $S$, of the point of contact to the above envelope. These can be written

$$\lambda \cos\alpha, \quad \lambda \sin\alpha, \quad \mu,$$

where $\mu$ must have the value (42) and $\lambda$ and $\alpha$ are to be determined. For this we make use of (40) and have only to express the condition that $d\zeta$ is zero. This gives the two equations

$$\left.\begin{aligned} &\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u} + \lambda \cos\alpha = 0, \\ &\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v} + \lambda \sin\alpha = 0, \end{aligned}\right\} \tag{51'}$$

which can be written

$$\frac{\partial}{\partial u}(A^2 + C^2) + 2\lambda A \cos\alpha = 0, \quad \frac{\partial}{\partial v}(A^2 + C^2) + 2\lambda C \sin\alpha = 0.$$

As in the case of equations (51), the above are compatible, so that we can determine $\lambda$ and $\alpha$ at once, and then the surface completely.

The envelope of this plane is parallel to $S$ when $\lambda$ is zero, which is possible only in case $A^2 + C^2$ is constant, that is, only when $S$ is pseudospherical or the parallel of such a surface. This could be seen also from the fact that $\mu$ must be zero, which leads to this particular case.

It has been seen that when $S$ is a pseudospherical surface, the lines joining corresponding points on $S$ and $S_1$ form a normal congruence. We inquire whether this is a general property for $A$-surfaces.

If we denote by $\psi$ the angle which this line forms with its projection on the tangent plane to $S$, it follows from the expressions for $\lambda$ and $\mu$ that

$$\sin\psi = \frac{-A\sin\omega + C\cos\omega}{\sqrt{A^2+C^2}}, \quad \cos\psi = \frac{A\cos\omega + C\sin\omega}{\sqrt{A^2+C^2}}. \tag{52}$$

Denote by $\rho$ the distance, measured along this line, from $M$ to the point $m$ of intersection with a normal surface; then the coordinates of $m$ with reference to the moving axes are

$$\rho\cos\psi\cos\theta, \quad \rho\cos\psi\sin\theta, \quad \rho\sin\psi.$$

The expressions for the displacements of $m$ are readily found in a manner similar to (40). If we note that the direction-cosines of the line are

$$\cos\psi\cos\theta, \quad \cos\psi\sin\theta, \quad \sin\psi,$$

the condition that the displacement be normal to this line can be reduced to

$$d\rho + A\cos\theta\cos\psi du + C\sin\theta\cos\psi dv = 0,$$

which can be replaced by the two equations

$$\frac{\partial\rho}{\partial u} + A\cos\theta\cos\psi = 0, \quad \frac{\partial\rho}{\partial v} + C\sin\theta\cos\psi = 0.$$

Expressing the condition of integrability and in the reduction making use of (52) and (6) we get

$$(A\sin\omega - C\cos\omega)\Big[C^3\sin\theta\frac{\partial}{\partial u}\log\sqrt{A^2+C^2} \\ + A^3\cos\theta\frac{\partial}{\partial v}\log\sqrt{A^2+C^2} + AC\sin\theta\cos\theta\Big] = 0,$$

when the first factor is equated to zero, we get pseudospherical surfaces, as we have seen before. It is evident that the second factor doesn't vanish in general. But there are exceptional cases when it is satisfied. Thus, it will be found later that there are certain $A$-surfaces which admit a point for a transform, in which case all the above lines meet in this point and hence are cut at right angles by a family of concentric spheres.

It is important to notice that the expressions (42) for $\lambda$ and $\mu$ do not involve $\theta$. Hence, the points of the transforms of $S$ corresponding to $M$ lie on a circle of radius $\lambda$ and whose plane is parallel to the tangent plane at $M$ and at a distance $\mu$ from it. Moreover, this circle cuts the infinity of transforms of $S$ orthogonally. Hence, these circles form a cyclic system and the axes of the circles are the normals to $S$. And it is well known that these normals form the only normal cyclic congruences.* Hence:

*The circles of a cyclic system whose congruence of axes is normal to a family of parallel surfaces $S$ are the loci of the points on the transforms of $S$ corresponding to the points of intersections of the normals with $S$.*

For the circles to be equal, $\lambda$ must be constant. Preceding results enable us to state this theorem:†

*When the circles of a cyclic system are equal and the congruence of axes is normal to a family of surfaces, the latter are the parallels of a pseudospherical surface. Moreover, the circles lie in the tangent planes to the latter, and have the points of contact for centers.*

In consequence of a preceding result, we have the theorem:

*The planes of the circles of a cyclic system, whose axes form a normal congruence, envelope an A-surface with the same representation of the lines of curvature as the surfaces orthogonal to the congruence. When these latter surfaces are the parallels of a pseudospherical surface, and only in this case, the above envelope belongs to this family of surfaces.*

## §6.—*Surfaces of Bianchi and their Transformations.*

In the introduction we have given Bianchi's definition of these surfaces and shall now determine their analytical expressions in the manner which he has suggested. With him we shall refer to them throughout the discussion as the surfaces $\Sigma$.

If we denote by $p$ and $q$ respectively, the distance from the origin upon the tangent plane and one-half the square of the distance to the point of contact, these surfaces are characterized by the equation*

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* Bianchi, Lezioni, p. 333; Ger. trans., p. 352.

† L. c., p. 332; Ger. trans., p. 351.

$$2q + k - (\rho_1 + \rho_2)p + \rho_1\rho_2 = 0, \tag{53}$$

where $\rho_1, \rho_2$ are the principal radii of curvature and $k$ is a constant, which is positive, negative or zero according as $\Sigma$ is of the elliptic, hyperbolic or parabolic type. The corresponding fixed sphere has its center at the origin and its radius $\sqrt{\pm k}$. It is important to remark that all the parallels of a surface $\Sigma$ are surfaces of Bianchi of the same type, since the centers of curvature are the same for all parallel surfaces.

Let $S$ be the pseudospherical surface with the linear element (21). The direction-cosines of the tangents to the lines of curvature, denoted by $X_1$, $Y_1$, $Z_1$; $X_2$, $Y_2$, $Z_2$, have the following expressions:

$$X_1 = \frac{1}{\cos\omega}\,\frac{\partial x}{\partial u}, \quad X_2 = \frac{1}{\sin\omega}\,\frac{\partial x}{\partial v}, \tag{54}$$

and similarly for $Y_1, \ldots\ldots, Z_2$. Recalling our previous definition of the positive directions of these lines, we can deduce the following:†

$$\left.\begin{aligned} &\frac{\partial X_1}{\partial u} = \frac{\partial \omega}{\partial v} X_2 - \sin\omega X, && \frac{\partial X_2}{\partial u} = -\frac{\partial \omega}{\partial v} X_1 \qquad\quad, && \frac{\partial X}{\partial u} = -\sin\omega X_1, \\ &\frac{\partial X_1}{\partial v} = \frac{\partial \omega}{\partial v} X_2 \qquad\qquad\quad, && \frac{\partial X_2}{\partial v} = -\frac{\partial \omega}{\partial u} X_1 + \cos\omega X, && \frac{\partial X}{\partial v} = -\cos\omega X_2, \end{aligned}\right\} \tag{55}$$

and similarly for the $Y$'s and $Z$'s.

Denote by $S_1'$ the transform by the particular integral $\theta_1$ of the pseudospherical surface $S$ with the linear element (21). Then the linear element of $S_1'$ has the form

$$ds_1'^2 = \cos^2\theta_1\, du_2 + \sin^2\theta_1\, dv^2,$$

which, by means of (46) and (50), can be put in the form

$$ds_1'^2 = d\alpha_1^2 + e^{2\alpha_1} d\gamma_1^2.$$

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* L. c., p. 347.

† Bianchi, Lezioni, p. 94; Ger. trans., p. 94.

If, now, the method of Weingarten* for the determination of surfaces satisfying equations of the general form (53), be applied to $S_1'$, we get for the coordinates, with respect to fixed rectangular axes, of the new surfaces

$$\xi = e^{-\alpha_1}\left(\frac{\partial x_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial x_1'}{\partial \gamma_1}\right), \quad \eta = e^{-\alpha_1}\left(\frac{\partial y_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial y_1'}{\partial \gamma_1}\right), \quad \zeta = e^{-\alpha_1}\left(\frac{\partial z_1'}{\partial \alpha_1} - \gamma_1 \frac{\partial z_1'}{\partial \gamma_1}\right).$$

But the coordinates of $S_1'$ with reference to these axes are

$$x_1' = x + \cos\theta_1 X_1 + \sin\theta_1 X_2,$$

and analogous expressions for $y_1'$ and $z_1'$. When these values are substituted in the above and one takes account of (46), (50) and (55), one gets

$$\xi = e^{-\alpha_1}(\cos\theta_1 X_1 + \sin\theta_1 X_2) + \gamma_1 X, \tag{56}$$

and similarly for $\eta$ and $\zeta$. From this we have by differentiation

$$\left.\begin{aligned} \frac{\partial \xi}{\partial u} &= -(e^{-\alpha_1}\cos\omega - \gamma_1 \sin\omega) X_1, \\ \frac{\partial \xi}{\partial v} &= -(e^{-\alpha_1}\sin\omega + \gamma_1 \cos\omega) X_2, \end{aligned}\right\} \tag{57}$$

from which it is seen that the tangent planes to $\Sigma$ and $S$ are parallel. Moreover, these equations show that the parametric system on $\Sigma$ is orthogonal, and being represented on the sphere by an orthogonal system, they are the lines of curvature for $\Sigma$, which shows that the latter is an $A$-surface. From (57) we have

$$A' = (e^{-\alpha_1}\cos\omega - \gamma_1 \sin\omega), \quad C' = -(e^{-\alpha_1}\sin\omega + \gamma_1\cos\omega); \tag{58}$$

it is readily found that these values satisfy the conditions (6). From (56) we find

$$2q = e^{-2\alpha_1} + \gamma_1^2, \quad p = \gamma_1,$$

and from (58),

$$\rho_1 = \gamma_1 - e^{-\alpha_1}\cot\omega, \quad \rho_2 = \gamma_1 + e^{-\alpha_1}\tan\omega.$$

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* C. R. March 23, 1891, and March 13, 1893.

When these values are substituted in (53), it is found that $k$ is zero, so that the surface defined by (56) is a surface of Bianchi of the parabolic type. Moreover, each particular integral of equations (38) gives a new surface of Bianchi of this type.

We pass now to the determination of the surfaces of Bonnet, which are the transforms of the above surface $\Sigma$. From (58) and (42) we have for the corresponding values of $\lambda$ and $\mu$,

$$\lambda = -e^{-\alpha_1}, \quad \mu = -\gamma_1,$$

so that the coordinates of the transform are

$$\xi_1 = \xi - e^{-\alpha_1}(\cos\theta X_1 + \sin\theta X_2) - \gamma_1 X,$$

and similarly for $\eta_1$ and $\zeta_1$, where $X_1, Y_1, Z_1$; $X_2, Y_2, Z_2$ are the direction-cosines of the tangents to the lines of curvature of $\Sigma$, and since these are parallel to the corresponding tangents for $S$, they have the values given by (54) and (55). When the value of $\xi$ from (56) is substituted in the above, the latter becomes

$$\xi_1 = e^{-\alpha_1}\left[(\cos\theta_1 - \cos\theta) X_1 + (\sin\theta_1 - \sin\theta) X_2\right]. \tag{59}$$

From this and similar expressions for $\eta_1$ and $\zeta_1$, it is evident that when $\Sigma$ is transformed by means of the same angle $\theta_1$ which is used in its definition, the transform $\Sigma_1$ reduces to a point, namely, the origin in the above system of coordinates. For the other transforms we have from (59),

$$\left.\begin{aligned} \frac{\partial\xi_1}{\partial u} &= e^{-\alpha_1}(\cos\theta_1 - \cos\theta)[X_1 \cos\omega \cos\theta + X_2 \cos\omega \sin\theta - X\sin\omega], \\ \frac{\partial\xi_1}{\partial v} &= e^{-\alpha_1}(\sin\theta_1 - \sin\theta)[X_1 \sin\omega \cos\theta + X_2 \sin\omega \sin\theta + X\cos\omega]. \end{aligned}\right\} \tag{60}$$

From these we get for the coefficients of the linear element of $\Sigma_1$,

$$A_1 = e^{-\alpha_1}(\cos\theta_1 - \cos\theta), \quad C_1 = e^{-\alpha_1}(\sin\theta_1 - \sin\theta). \tag{61}$$

From (59) we have

$$\Sigma \xi_1 X = 0,$$

that is, the radii vectores of the surfaces $\Sigma_1$ are parallel to the corresponding tangent planes to $\Sigma$.

The preceding investigations show that the normals to $\Sigma_1$ have for direction-cosines the expressions

$$\sin\theta X_1-\cos\theta X_2,\quad \sin\theta Y_1-\cos\theta Y_2,\quad \sin\theta Z_1-\cos\theta Z_2,$$

so that from (59) we have, for the distances from the origin to the tangent planes to $\Sigma_1$ and half the square of the distance to the point of contact, the expression

$$p_1=-e^{-\alpha_1}\sin(\theta_1-\theta),\quad q_1=e^{-2\alpha_1}[1-\cos(\theta_1-\theta)].$$

Again, the principal radii of curvature have the expressions

$$\rho_1=\frac{e^{-\alpha_1}(\cos\theta_1-\cos\theta)}{\sin\theta},\quad \rho_2=\frac{e^{-\alpha_1}(\sin\theta-\sin\theta_1)}{\sin\theta}.$$

When these values are substituted in (53), the latter is satisfied identically if $k$ is zero. Hence:

*All the transforms of a surface of Bianchi of the parabolic type are surfaces of Bianchi of the same type and one reduces to a point.* From the previous remarks about the associated cyclic systems, it follows that all the circles must pass through this point, so that—

*The circles of the cyclic system whose axes are the normals to a surface of Bianchi of the parabolic type, pass through a point and the infinity of surfaces orthogonal to these circles are surfaces of Bianchi of the same type.*

We have seen that the transforms of these surfaces $\Sigma_1$ by the angle $\omega+\pi$ are one surface, namely, the envelope of the plane parallel to the tangent plane to $\Sigma$ and at a distance $\mu$ from it. Moreover, this is also the envelope of the planes of the circles of the cyclic system associated with $\Sigma$. The function $\lambda_1$ for all the transformations of the surfaces $\Sigma_1$ is found from (61) by (42) to have the value

$$\lambda_1=e^{-\alpha_1}[\cos(\theta-\theta_1)-1].$$

When this value and those for $A_1$ and $C_1$, given by (61), are substituted in formulæ similar to (44) and $\omega+\pi$ is the angle of transformation, we find that the coefficients $A''$, $C''$ of the linear element of this transform take the value zero, so that the transform is a point. We know that the coordinates of this point with respect to the trihedron of $\Sigma$ are

$$\lambda \cos \theta, \quad \lambda \sin \theta, \quad \mu,$$

where $\lambda$ and $\theta$ are given by (51′) and $\mu$ is $-\gamma_1$. If the values for $A$ and $C$ from (58) are substituted in (51′), it is found that

$$\lambda = - e^{-a_1}, \quad \theta = \theta_1 .$$

This shows that the original circle for $\Sigma$ passes through this point, so that we have the following theorem:

*The surface of the same class as* $\Sigma$, *which is the transform of all the surfaces* $\Sigma_1$, *is a point and the same one through which pass all the circles of the cyclic system associated with* $\Sigma$.

But every solution of the system (38) gives a surface $\Sigma$, and the above point is the origin in every case. Hence the theorem:

*Among all the normal cyclic congruences, whose developables have the same spherical representation, there is an infinity whose associated circles pass through a point, the same for all the congruences.*

Bianchi has shown that this theorem is true for all cyclic congruences.*

In a purely geometrical manner, Bianchi has established the following theorem concerning surfaces $\Sigma$: †

*If one take any surface* $\Sigma_1$ *of the parabolic type and a sphere* $S_0$ *with center at the origin, the circles normal to* $\Sigma_1$ *and which cut the sphere in diametrically opposite points or orthogonally admit an infinity of orthogonal surfaces which are in every case surfaces* $\Sigma$ *of the parabolic type, and the axes of these circles are cut normally by a family of surfaces* $\Sigma$ *of the elliptic type in the former case and of the hyperbolic type in the latter; when* $S_0$ *reduces to a point, the latter family is composed of surfaces of the parabolic type.*

In accordance with this theorem, Bianchi finds the analytical expressions for these surfaces. We shall recall his results briefly and then apply to these surfaces the above transformations.

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* Lezioni, p. 335; Ger. trans., p. 353.

† Annali, l. c., p. 367.

Bianchi takes for the surface $\Sigma_1$ of the parabolic type, one which is derived from the pseudospherical surface $S$ in a manner similar to the derivation of $\Sigma$ from $S_1'$. The coordinates of a point on $\Sigma_1$ are found to have the expressions

$$\xi_1 = (e^{\alpha} \cos\theta_1 + \beta \sin\theta_1) X_1 + (e^{\alpha} \sin\theta_1 - \beta\cos\theta_1) X_2,$$

and, similarly, for $\eta_1$ and $\zeta_1$; $\alpha$, $\beta$ are the functions given by (46) and $X_1 \ldots Z_2$ by (54).

We denote by $\xi_0$, $\eta_0$, $\zeta_0$ the coordinates of the center and by $R$ the radius of the circle, drawn as indicated in the above theorem, and so that the plane of the circle be parallel to the tangent plane to $S$; we have

$$\xi_0 = \xi_1 + R(\cos\theta_1 X_1 + \sin\theta_1 X_2),$$

whence

$$\xi_0 = [(e^{\alpha} + R)\cos\theta_1 + \beta\sin\theta_1] X_1 + [(e^{\alpha} + R)\sin\theta_1 - \beta\cos\theta_1] X_2,$$

and, similarly, for $\eta_0$ and $\zeta_0$. In accordance with the above definition of the surfaces, one must have

$$\Sigma\xi_0^2 = R^2 - k,$$

where $k$ is a constant, positive or negative according as the circles cut the sphere of radius $\sqrt{\pm k}$ in diametrically opposite points or orthogonally. This condition leads to

$$e^{\alpha} + R = \tfrac{1}{2}[e^{\alpha} - (\beta^2 + k)e^{-\alpha}],$$

so that the above expression for $\xi_0$ can be written

$$\xi_0 = [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\cos\theta_1 + \beta\sin\theta_1] X_1 \\ + [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\sin\theta_1 - \beta\cos\theta_1] X_2. \qquad (62)$$

The congruence of the axes of these circles is defined by the formula

$$\xi = \xi_0 + tX, \quad \eta = \eta_0 + tY, \quad \zeta = \zeta_0 + tZ.$$

For these formulæ to define a surface normal to the congruence, $t$ must satisfy the condition

$$\Sigma X d\xi = 0;$$

this gives for the determination of $t$, in consequence of (62), the equation

$$dt = [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\cos\theta_1 + \beta\sin\theta_1]\sin\omega du \\ - [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\sin\theta_1 - \beta\cos\theta_1]\cos\omega dv; \qquad (63)$$

the right-hand member can readily be shown, by means of (38) and (46), to be an exact differential. Hence, the normal surfaces are given by

$$\xi = [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\cos\theta_1 + \beta\sin\theta_1]X_1 \\ + [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\sin\theta_1 - \beta\cos\theta_1]X_2 + tX, \qquad (64)$$

and analogous expressions for $\eta$ and $\xi$. From (64) one gets

$$\left.\begin{aligned} \frac{\partial\xi}{\partial u} &= [\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\cos\omega + t\sin\omega]X_1, \\ \frac{\partial\xi}{\partial v} &= [\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\sin\omega - t\cos\omega]X_2, \end{aligned}\right\} \qquad (65)$$

from which it is readily seen that the parametric system on these surfaces is orthogonal, and since these lines have the spherical representation (4), it follows that the surfaces defined by (64) are surfaces of Bonnet. From (65) we have

$$\left.\begin{aligned} A &= \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\cos\omega + t\sin\omega, \\ C &= \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\sin\omega - t\cos\omega, \end{aligned}\right\} \qquad (66)$$

which values are readily found to satisfy the condition (6). Bianchi[*] shows that the equation (53) is satisfied by these surfaces and $k$ has the same meaning in both. We proceed now to find the transforms of these surfaces.

In accordance with the formulæ (42), one has at once

$$\lambda = \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}, \quad \mu = -t, \qquad (67)$$

and by (44),

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\cos\theta_1 + \beta\sin\theta_1 + \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\cos\theta, \\ C_1 &= \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)e^{-\alpha}\}\sin\theta_1 - \beta\cos\theta_1 + \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}\sin\theta. \end{aligned}\right\} \qquad (68)$$

We can find directly the coordinates $\xi_1, \eta_1, \zeta_1$, with reference to fixed axes, of these surfaces. By means of (67), we have

$$\xi_1 = \xi + \tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)e^{-\alpha}\}[X_1\cos\theta + X_2\sin\theta] - tX,$$

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[*] Annali, l. c., p. 368.

and analogous expressions for $\eta_1$ and $\zeta_1$. If the value for $\xi$ from (64) be put in the right-hand member, this becomes

$$\begin{aligned}\xi_1 = {} & [\tfrac{1}{2}e^{\alpha}(\cos\theta_1+\cos\theta)+\tfrac{1}{2}(\beta^2+k)e^{-\alpha}(\cos\theta-\cos\theta_1)+\beta\sin\theta_1]X_1 \\ & +[\tfrac{1}{2}e^{\alpha}(\sin\theta_1+\sin\theta)+\tfrac{1}{2}(\beta^2+k)e^{-\alpha}(\sin\theta-\sin\theta_1)-\beta\cos\theta_1]X_2. \quad (69)\end{aligned}$$

When in particular, $\Sigma$ is transformed by the angle $\theta_1$, the transform is given by

$$\xi_1 = [e^{\alpha}\cos\theta_1+\beta\sin\theta_1]X_1+[e^{\alpha}\sin\theta_1-\beta\cos\theta_1]X_2,$$

and similar expressions in $\eta_1$ and $\zeta_1$. But this is the surface $\Sigma_1$ from which we started, and hence is of the parabolic type. Moreover, from the defining theorem of Bianchi, it follows that *all the surfaces defined by* (69) *are surfaces* $\Sigma$ *of the parabolic type*, for they form the infinite family of surfaces which cut orthogonally the circles whose axes are normal to the surfaces $\Sigma$ defined by (64). When $k$ is zero, one of these surfaces must reduce to a point, namely, the origin. To find the value of $\theta$, corresponding we have only to equate $A_1$ and $C_1$ to zero, which gives

$$\begin{aligned}(e^{\alpha}+\beta^2e^{-\alpha})\cos\theta &= (\beta^2e^{-\alpha}-e^{\alpha})\cos\theta_1-2\beta\sin\theta_1, \\ (e^{\alpha}+\beta^2e^{-\alpha})\sin\theta &= (\beta^2e^{-\alpha}-e^{\alpha})\sin\theta_1+2\beta\cos\theta_1,\end{aligned}$$

and when these values are put in (69), we get the desired result.

Again, since the surfaces $\Sigma_1$ are of the parabolic type, the circles of their associate cyclic system must pass through a point. To find this point, we have only to determine the value of $\phi$ which makes $\Sigma_2$ reduce to a point. From (68) we have, for $\lambda_1$ and $\mu_1$,

$$\begin{aligned}\lambda_1 &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}+\tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\cos(\theta_1-\theta)+\beta\sin(\theta_1-\theta), \\ \mu_1 &= \tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\sin(\theta_1-\theta)-\beta\cos(\theta_1-\theta),\end{aligned}$$

so that if we determine $A_2$ and $C_2$ from expressions similar to (44), we get

$$A_2=\lambda_1(\cos\omega+\cos\phi), \quad C_2=\lambda_1(\sin\omega+\sin\phi).$$

From this it is seen that $\Sigma_2$ is a point, when any of the surfaces defined by (69) are transformed by the angle $\omega+\pi$. On this account the plane through this point is parallel to the tangent plane to $\Sigma$ and, consequently, the length $\lambda_1$ is measured along the line whose direction-cosines are

$$-(X_1 \cos\theta + X_2 \sin\theta), \quad -(Y_1 \cos\theta + Y_2 \sin\theta), \quad -(Z_1 \cos\theta + Z_2 \sin\theta),$$

and $\mu_1$ along the line whose direction-cosines are

$$X_1 \sin\theta - X_2 \cos\theta, \quad Y_1 \sin\theta - Y_2 \cos\theta, \quad Z_1 \sin\theta - Z_2 \cos\theta. \tag{70}$$

Hence the coordinates of this particular surface $\Sigma_2$ are

$$\xi_2 = \xi_1 - \lambda_1 (X_1 \cos\theta + X_2 \sin\theta) + \mu_1 (X_1 \sin\theta - X_2 \cos\theta),$$

and similar ones for $\eta_2$ and $\zeta_2$. When the values for $\xi_1$, $\eta_1$, $\zeta_1$ are substituted from (69), they reduce to zero, so that $\Sigma_2$ is the origin. Moreover, this is entirely independent of the angle $\theta$ by which the surfaces $\Sigma$ were transformed. Hence:

*The circles of the cyclic system associated with the transform of a surface of Bianchi* (64) *pass through the same point as the circles of the cyclic system associated with a surface $\Sigma$ of the parabolic type, with the same spherical representation of its lines of curvature as the original surface.*

From (68), (69) and (70), we get, for the quantities $p_1$, $q_1$, $\rho_1$ and $\rho_2$ for the surface $\Sigma_1$ the values

$$\begin{aligned} 2q_1 &= \tfrac{1}{2}\left[e^{2\alpha} + e^{-2\alpha}(\beta^2 + k)^2\right] + \tfrac{1}{2}\left[e^{2\alpha} - e^{-2\alpha}(\beta^2 + k)^2\right] \cos(\theta_1 - \theta) + \beta^2 \\ &\qquad + \beta\left[e^{\alpha} + e^{-\alpha}(\beta^2 + k)\right] \sin(\theta_1 - \theta), \\ p_1 &= -\tfrac{1}{2}\left[e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right] \sin(\theta_1 - \theta) + \beta \cos(\theta_1 - \theta), \\ \rho_1 &= \frac{A_1}{\sin\theta}, \quad \rho_2 = -\frac{C_1}{\cos\theta}. \end{aligned}$$

Between these functions there obtains the relation

$$2q_1 - p_1(\rho_1 + \rho_2) + \rho_1\rho_2 = 0,$$

which shows that the surfaces $\Sigma_1$ are parabolic and that the associated circles pass through the origin, affording a verification of the preceding.

## §7.—*General Transformations of A-surfaces.*

We proceed now to the definition of more general transformations of the $A$-surfaces. We consider the envelope of the plane which cuts the tangent plane to such a surface under a constant angle $\sigma$ and along the line through the point of tangency which makes an angle $\theta$ with the tangent to the line of curvature $v = \text{const.}$ through the point. It will be found that for determinate values of $\theta$ this envelope is an $A$-surface.

As before we consider $S$ with reference to the moving trihedron; denote by $M$ the point of tangency on $S$ and $M_1$ the corresponding point on the envelope $S_1$. In the enveloping plane we drop from $M_1$ a perpendicular upon the line of intersection of this plane with the tangent plane to $S$, and denote its length, by $\mu$. Further, we let $\lambda$ denote the length from $M$ to the foot of this perpendicular. Then the coordinates of $M_1$ with respect to the moving axes have the expressions

$$x = \lambda \cos\theta - \mu \cos\sigma \sin\theta, \quad y = \lambda \sin\theta + \mu \cos\sigma \cos\theta, \quad z = \mu \sin\sigma. \tag{80}$$

The projections upon the axes of a displacement of $M_1$ have the values[*] from (5)

$$\begin{aligned}
&dx + (A + \mu \sin\omega \sin\sigma)\, du - (\lambda \sin\theta + \mu \cos\sigma \cos\theta)\left(\frac{\partial\omega}{\partial v}\, du + \frac{\partial\omega}{\partial u}\, dv\right), \\
&dy + (C - \mu \cos\omega \sin\sigma)\, dv + (\lambda \cos\theta - \mu \cos\sigma \sin\theta)\left(\frac{\partial\omega}{\partial v}\, du + \frac{\partial\omega}{\partial u}\, dv\right). \\
&dz + \cos\omega\,(\lambda \sin\theta + \mu \cos\sigma \cos\theta)\, dv - \sin\omega\,(\lambda \cos\theta - \mu \cos\sigma \sin\theta)\, du.
\end{aligned}$$

We project these three lengths upon the three orthogonal directions formed by the line, $MP$, of the intersection of the two planes, the line, $MQ$, in the tangent plane perpendicular to this and the normal, $MN$, to the surface. These projections are readily found to be

$$\left.\begin{aligned}
&d\lambda - \mu \cos\sigma\, d\theta + A \cos\theta\, du + C \sin\theta\, dv - \mu \cos\sigma \left(\frac{\partial\omega}{\partial v}\, du + \frac{\partial\omega}{\partial u} dv\right) \\
&\qquad\qquad + \mu \sin\sigma\,(\sin\omega \cos\theta\, du - \cos\omega \sin\theta\, dv), \\
&\lambda\, d\theta + \cos\sigma\, d\mu - A \sin\theta\, du + C \cos\theta\, dv + \lambda \left(\frac{\partial\omega}{\partial v}\, du + \frac{\partial\omega}{\partial u}\, dv\right) \\
&\qquad\qquad - \mu \sin\sigma\,(\sin\omega \sin\theta\, du + \cos\omega \cos\theta\, dv), \\
&\sin\sigma\, d\mu + \cos\omega\,(\lambda \sin\theta + \mu \cos\sigma \cos\theta) - \sin\omega\,(\lambda \cos\theta - \mu \cos\sigma \sin\theta).
\end{aligned}\right\} \tag{81}$$

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[*] Darboux, Leçons, Vol. 2, p. 385.

The direction-cosines of the given plane with respect to the lines *MP, MQ, MN* are evidently

$$0, \quad -\sin\sigma_1, \; \cos\sigma.$$

Consequently, the necessary and sufficient condition that this plane envelope $S_1$ is that these functions satisfy the equation

$$\begin{aligned}&\left(\lambda\sin\sigma\frac{\partial\theta}{\partial u}-A\sin\theta\sin\sigma+\lambda\frac{\partial\omega}{\partial v}\sin\sigma+\lambda\cos\theta\sin\omega\cos\sigma-\mu\sin\omega\sin\theta\right)du\\&\quad+\left(\lambda\sin\sigma\frac{\partial\theta}{\partial v}+C\sin\sigma\cos\theta+\lambda\frac{\partial\omega}{\partial u}\sin\sigma-\lambda\sin\theta\cos\omega\cos\sigma\right.\\&\quad\left.-\mu\cos\omega\cos\theta\right)dv=0.\end{aligned}\tag{82}$$

We consider first the case where $S$ is pseudospherical, so that we can take

$$A=\cos\omega, \qquad C=\sin\omega.\tag{83}$$

When these values are substituted in (82), we get, by equating to zero, the coefficients of $du$ and $dv$, the following equations of condition:

$$\left.\begin{aligned}\lambda\sin\sigma\left(\frac{\partial\theta}{\partial u}+\frac{\partial\omega}{\partial v}\right)&=\sin\sigma\sin\theta\cos\omega-\lambda\cos\sigma\cos\theta\sin\omega+\mu\sin\theta\sin\omega,\\ \lambda\sin\sigma\left(\frac{\partial\theta}{\partial v}+\frac{\partial\omega}{\partial u}\right)&=-\sin\sigma\cos\theta\sin\omega+\lambda\cos\sigma\sin\theta\cos\omega+\mu\cos\theta\cos\omega.\end{aligned}\right\}\tag{84}$$

Taking the derivative of the first with respect to $v$ and the second with respect to $u$, subtracting and noting that $\omega$ satisfies equation (7), we have

$$\begin{aligned}&(\sin\sigma\sin\theta\cos\omega+\mu\sin\theta\sin\omega)\frac{\partial\log\lambda}{\partial v}\\&\qquad\qquad+(\sin\sigma\cos\theta\sin\omega-\mu\cos\theta\cos\omega)\frac{\partial\log\lambda}{\partial u}\\&\quad+\frac{(\sin^2\sigma-\lambda^2-\mu^2)\sin\omega\cos\omega}{\lambda\sin\sigma}-\sin\theta\sin\omega\frac{\partial\mu}{\partial v}+\cos\theta\cos\omega\frac{\partial\mu}{\partial u}=0.\end{aligned}\tag{85}$$

If, in turn, we take the derivative of the first of (84) with respect to $u$ and the second with respect to $v$ and subtract and *then make the hypothesis that $\theta$ is a solu-*

*tion of equation* (7), we are brought to the equation

$$(\sin\sigma\cos\theta\sin\omega - \mu\cos\theta\cos\omega)\frac{\partial\log\lambda}{\partial v}$$

$$+ (\sin\sigma\sin\theta\cos\omega + \mu\sin\theta\sin\omega)\frac{\partial\log\lambda}{\partial u}$$

$$+ \frac{(\lambda^2 - \mu^2 - \sin^2\sigma)\sin\theta\cos\theta + \lambda\mu\cos\sigma(\cos^2\theta - \sin^2\theta)}{\lambda\sin\sigma}$$

$$+ \cos\theta\cos\omega\frac{\partial\mu}{\partial v} - \sin\theta\sin\omega\frac{\partial\mu}{\partial u} = 0. \tag{86}$$

If, now, we have given two functions $\lambda$ and $\mu$ satisfying equations (85) and (86) and these values be substituted in (84), the latter will define a function $\theta_1$ involving an arbitrary constant, which satisfies equation (7) and such that the surface defined by (80) is the envelope of this plane of angle $\sigma$ with the tangent plane to $S$.

Consider the case where $\lambda$ and $\mu$ are constant. Equations (85) and (86) reduce to

$$(\sin^2\sigma - \lambda^2 - \mu^2)\sin\omega\cos\omega = 0,$$
$$(\lambda^2 - \mu^2 - \sin^2\sigma)\sin\theta\cos\theta + \lambda\mu\cos\sigma(\cos^2\theta - \sin^2\theta) = 0.$$

Since $\lambda$ and $\mu$ are independent of $\theta$, these may be replaced by

$$\sin^2\sigma - \lambda^2 - \mu^2 = 0, \quad \lambda^2 - \mu^2 - \sin^2\sigma = 0, \quad \lambda\mu = 0,$$

from which it follows that

$$\lambda^2 = \sin^2\sigma, \quad \mu = 0. \tag{87}$$

If we take

$$\lambda = \sin\sigma, \quad \mu = 0, \tag{87'}$$

the equations (84) become

$$\left.\begin{aligned}\sin\sigma\left(\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v}\right) &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega,\\ \sin\sigma\left(\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega.\end{aligned}\right\} \tag{88}$$

When the values from (87′) and (83) are substituted in (81) and in the reduction use is made of equations (88), one gets for the projections of the displacement of $M_1$,

$$\left.\begin{aligned} &\cos\omega\cos\theta du+\sin\omega\sin\theta dv,\\ &-\cos\sigma(\sin\omega\ \cos\theta du-\cos\omega\ \sin\theta dv),\\ &-\sin\sigma\ (\sin\omega\ \cos\theta du-\cos\omega\ \sin\theta dv),\end{aligned}\right\} \tag{89}$$

and from these we get for the linear element of $S_1$ the expression

$$ds_1^2=\cos^2\theta du^2+\sin^2\theta dv^2. \tag{89'}$$

These are the same results which Darboux gets in his study of the transformations of pseudospherical surfaces.* From them he shows that the parametric curves on $S_1$ are its lines of curvature so that $S_1$ is the pseudospherical surface corresponding to the solution $\theta$ of equation (7). This transformation, which is equivalent to finding the locus of points on the lines tangent to the curves on $S_1$ given by a solution of the system (88), and at a distance $\sin\sigma$, was discovered by Bäcklund and has since that time been called by his name. It is evident that the transformations which we are discussing are a generalization of these transformations of Bäcklund.

The other solution, $\lambda=-\sin\sigma$ of (87) offers nothing new as it merely gives a Bäcklund transformation of different angles.

We return now to the case of consideration of $A$-surfaces in general and discuss the case where $\theta$ satisfies equations (88). For this choice of $\theta$, the equation (82) can be replaced by

$$\left.\begin{aligned} A\sin\sigma&=\lambda\cos\omega-\mu\sin\omega,\\ C\sin\sigma&=\lambda\sin\omega+\mu\cos\omega,\end{aligned}\right\} \tag{90}$$

from which it follows that

$$\left.\begin{aligned} \lambda&=(A\ \cos\omega+C\sin\omega)\sin\sigma,\\ \mu&=(-A\sin\omega+C\cos\omega)\sin\sigma.\end{aligned}\right\} \tag{91}$$

Substituting these values in (81), these projections of the displacement of $M_1$ become

$$\left.\begin{aligned} &A_1\cos\omega\ du+C_1\sin\omega\ dv,\\ &-\cos\sigma\ (A_1\sin\omega\ du-C_1\cos\omega\ dv),\\ &-\sin\sigma\ (A_1\sin\omega\ du-C_1\cos\omega\ dv),\end{aligned}\right\} \tag{92}$$

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* Darboux, Leçons, Vol. 3, p. 435.

where we have put for brevity

$$\left.\begin{aligned} A_1 &= \sin\sigma\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + \lambda\frac{\cos\theta}{\sin\sigma} - \mu\frac{\cos\sigma\sin\theta}{\sin\sigma}, \\ C_1 &= \sin\sigma\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) + \lambda\frac{\sin\theta}{\sin\sigma} + \mu\frac{\cos\sigma\cos\theta}{\sin\sigma}. \end{aligned}\right\} \tag{93}$$

From this we find for the linear element of the envelope $S_1$

$$ds_1^2 = A_1^2 du^2 + C_1^2 dv^2. \tag{94}$$

By a system of reasoning similar to that followed in the case where $\sigma$ was $\pi/2$ we can show that the lines $u = \text{const.}$, $v = \text{const.}$ are lines of curvature and hence $S_1$ is an $A$-surface. Hence the theorem:

*Given an A-surface $S$, corresponding to a solution $\omega$ of equation* (7), *and given also a solution $\theta$ of equations* (88) *for any angle $\sigma$. The plane which cuts the tangent plane to $S$ under the angle $\sigma$ and along the line through the point of contact which makes the angle $\theta$ with the tangent to the line of curvature $v = \text{const.}$ is an A-surface with the spherical representation*

$$\sin^2\theta du^2 + \cos^2\theta dv^2. \tag{95}$$

When $\sigma = \pi/2$ we call the transformation *complementary.*

As in the case where $\sigma$ was a right angle, the expressions for $\mu$ and $\lambda$ are independent of $\theta$, so that the points of tangency with their envelopes of all planes through a given point $M$ of $S$ lie on circles, each angle $\sigma$ giving rise to a separate circle, and the planes of these circles are parallel to the tangent plane to $S$ and approach it as $\sigma$ approaches zero. Hence with every $A$-surface there is associated not only a cyclic system of circles, but an infinity of systems of circles, the circles of each system being cut under the same angle by a family of $A$-surfaces. But, as we have seen, these are the only cyclic systems whose circles have axes forming a normal congruence; consequently, we have the theorem:

*The lines of a normal cyclic congruence are the axes of an infinity of circles which are cut under different constant angles by families of surfaces.*

From the definition of $\mu$ and its expression (91) it is evident that the planes of the circles of these systems are at the distance

$$(-A\sin\omega + C\cos\omega)\sin^2\sigma$$

from the tangent plane to $S$, which shows that the distance decreases with $\sigma$.

We have found (20) that the linear element of a surface parallel to a pseudospherical surface has the linear element

$$ds^2 = (\cos\omega - a\sin\omega)^2\,du^2 + (\sin\omega + a\cos\omega)\,dv^2.$$

If these values for $A$ and $C$ be substituted in (91), we get

$$\lambda = \sin\sigma, \qquad \mu = a\sin\sigma.$$

When all these values are substituted in (93), the latter become

$$A_1 = \cos\theta - a\cos\sigma\sin\theta_1 \qquad C_1 = \sin\theta + a\cos\sigma\cos\theta.$$

A comparison of these expressions with the coefficients of the above linear element makes it clear that the transform of a parallel, $S$, of a pseudospherical surface $\Sigma$ is parallel to the transform $\Sigma$ by means of the same angles $\theta$ and $\sigma$. From the values of $A_1$ and $C_1$ it follows that the transform of $S$ is pseudospherical only when $\sigma$ is $\pi/2$.

As in the case of transformations with $\sigma$ a right angle, it is seen that when $S_1$ is found the complete determination of the other $A$-surfaces with the same spherical representation of their lines of curvature requires the integration of (47), where $\theta$ has the value given by the solution of equations (88).

If, with Bianchi,[*] we denote by $B_\sigma$ the Bäcklund transformation of angle $\sigma$, it can be shown, as Lie first pointed out, that

$$B_\sigma = L_\sigma B_{\frac{\pi}{2}} L_\sigma^{-1},$$

where $L_\sigma$ is given by (29). This result, which was established for transformation of pseudospherical surfaces, is just as valid for transformation of any $A$-surface, for in its derivation no use is made of the linear element of the surface.

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[*] Bianchi, Lezioni, p. 434; Ger. trans., 460.

Hence, all the transformations of surfaces which we have been discussing can be obtained by transformations of Lie and complementary transformations.

## §8.—*Theorem of Permutability for A-surfaces.*

Bianchi has established for the transformations of Bäcklund the *theorem of permutability* by means of which it can be shown that when one can find all the transforms $S_1$ of a given pseudospherical surface $S$, the transformations of these surfaces $S_1$ are given by algebraic processes and differentiation. This theorem is readily established when the surface is referred to the moving trihedron. After deducing it, we shall apply it to the theory of transformations of $A$-surfaces.

Let $S_1$ be the transform of $S$ corresponding to the angles $\theta_1$ and $\sigma_1$, and $S_2$ for the angles $\theta_2$, $\sigma_2$. Let $\phi$ be a solution of the system

$$\left.\begin{aligned} \sin\sigma_2\left(\frac{\partial\phi}{\partial u}+\frac{\partial\theta_1}{\partial v}\right) &= \sin\phi\cos\theta_1-\cos\sigma_2\cos\phi\sin\theta_1, \\ \sin\sigma_2\left(\frac{\partial\phi}{\partial v}+\frac{\partial\theta_1}{\partial u}\right) &= -\cos\phi\sin\theta_1+\cos\sigma_2\sin\phi\cos\theta_1, \end{aligned}\right\} \qquad (96)$$

and denote by $S_3$ the corresponding transform of $S_1$. It is evident that the projections on the tangents to the lines of curvature and the normal to $S_1$ of the line $M_1 M_3$ are

$$\sin\sigma_2\cos\phi, \quad \sin\sigma_2\sin\phi, \quad 0.$$

Again, if we denote by $X_1'$, $Y_1'$, $Z_1'$; $X_1''$, $Y_1''$, $Z_1''$, the direction-cosines of these tangents to the lines of curvature of $S_1$ with respect to the directions $MP$, $MQ$, $MR$ for $S$, we find from (89) and (89′),

$$\left.\begin{aligned} X_1',\ Y_1',\ Z_1' &= \cos\omega, \ -\cos\sigma_1\sin\omega, \ -\sin\sigma_1\sin\omega, \\ X_2',\ Y_2',\ Z_2' &= \sin\omega, \quad \cos\sigma_1\cos\omega, \quad \sin\sigma_1\cos\omega. \end{aligned}\right\} \qquad (98)$$

From these expressions and (97) it follows that the coordinates of $M_3$, with respect to the axes $MP$, $MQ$, $MN$, have the expressions

$$\xi=\sin\sigma_1+\sin\sigma_2\cos(\phi-\omega), \quad \eta=\sin\sigma_2\cos\sigma_1\sin(\phi-\omega),$$
$$\zeta=\sin\sigma_1\sin\sigma_2\sin(\phi-\omega).$$

Recalling the definition of *MP* and *MQ*, we find for the coordinates, $x_3$, $y_3$, $z_3$, of $M_3$ with respect to the axes of the moving trihedron

$$\left.\begin{aligned}x_3 &= \cos\theta_1 \sin\sigma_1 + \cos\theta_1 \sin\sigma_2 \cos(\phi - \omega) - \sin\theta_1 \sin\sigma_2 \cos\sigma_1 \sin(\phi - \omega),\\ y_3 &= \sin\theta_1 \sin\sigma_1 + \sin\theta_1 \sin\sigma_2 \cos(\phi - \omega) + \cos\theta_1 \sin\sigma_2 \cos\sigma_1 \sin(\phi - \omega),\\ z_3 &= \sin\sigma_1 \sin\sigma_2 \sin(\phi - \omega).\end{aligned}\right\} \quad (99)$$

We assume that there is a solution $\phi$ of equations (96) which satisfies the equations

$$\left.\begin{aligned}\sin\sigma_1\left(\frac{\partial\phi}{\partial u} + \frac{\partial\theta_2}{\partial v}\right) &= \sin\phi\cos\theta_2 - \cos\sigma_1\cos\phi\sin\theta_2,\\ \sin\sigma_1\left(\frac{\partial\phi}{\partial v} + \frac{\partial\theta_2}{\partial u}\right) &= -\cos\phi\sin\theta_2 + \cos\sigma_1\sin\phi\cos\theta_2.\end{aligned}\right\} \quad (100)$$

If $S_2$ is transformed by means of this same function $\phi$ and the angle $\sigma_1$, we find for the coordinates, denoted by $x_3'$, $y_3'$, $z_3'$, of the transform expressions which can be obtained from (99) by interchanging the subscripts 1 and 2. Then $z_3$ and $z_3'$ are equal; we assume further that $x_3$ and $x_3'$ are equal, and also $y_3$ and $y_3'$. It is necessary that we have

$$\begin{aligned}\cos\theta_1(x_3 - x_3') + \sin\theta_1(y_3 - y_3') &= 0,\\ \cos\theta_2(x_3 - x_3') + \sin\theta_2(y_3 - y_3') &= 0;\end{aligned}$$

and these equations are the sufficient condition for the equality of the coordinates, for otherwise $\sin(\theta_1 - \theta_2)$ is zero which is evidently impossible.

If the above expressions for $x_3$, $y_3$, $x_3'$, $y_3'$ are substituted in these equations they become

$$\left.\begin{aligned}&[\sin\sigma_1\cos(\theta_2 - \theta_1) - \sin\sigma_2]\cos(\phi - \omega)\\ &\quad - \sin\sigma_1\cos\sigma_2\sin(\theta_2 - \theta_1)\sin(\phi - \omega) = \sin\sigma_1 - \sin\sigma_2\cos(\theta_2 - \theta_1),\\ &[\sin\sigma_2\cos(\theta_2 - \theta_1) - \sin\sigma_1]\cos(\phi - \omega)\\ &\quad - \sin\sigma_2\cos\sigma_1\sin(\theta_2 - \theta_1)\sin(\phi - \omega) = \sin\sigma_2 - \sin\sigma_1\cos(\theta_2 - \theta_1).\end{aligned}\right\} \quad (101)$$

Solving these equations with respect to $\sin(\phi - \omega)$ and $\cos(\phi - \omega)$, one gets

$$\left.\begin{aligned}\sin(\phi - \omega) &= \frac{(\cos\sigma_2 - \cos\sigma_1)\sin(\theta_2 - \theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2 - \theta_1) + \cos\sigma_1\cos\sigma_2 - 1},\\ \cos(\phi - \omega) &= \frac{\sin\sigma_1\sin\sigma_2 + (\cos\sigma_1\cos\sigma_2 - 1)\cos(\theta_2 - \theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2 - \theta_1) + \cos\sigma_1\cos\sigma_2 - 1},\end{aligned}\right\} \quad (102)$$

which are readily found to satisfy the condition that the sum of the squares is equal to unity. From these we get

$$\tan\left(\frac{\phi-\omega}{2}\right)=\frac{\sin\left(\frac{\sigma_1+\sigma_2}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma_2}{2}\right)}\tan\left(\frac{\theta_1-\theta_2}{2}\right), \tag{103}$$

which is similar to the result of Bianchi*. As he has remarked, it is readily shown that this value of $\phi$ satisfies equations (96) and (100), so that the preceding hypotheses are consistent. Hence the transform of $S_1$ by $\phi$ and the angle $\sigma_2$ is the same pseudospherical surface as the transform of $S_2$ by $\phi$ and the angle $\sigma_1$, which is the so-called *theorem of permutability.*

Suppose then that one has the general solution of the system (88), say

$$\theta\,(u, v, \sigma, c).$$

Let $S_1'$ denote the transform of $S$ corresponding to the function $\theta\,(u, v, \sigma_1, c_1)$; then all the transforms of $S_1'$ are obtained by means of the function $\phi$ given by

$$\tan\frac{\phi-\omega}{2}=\frac{\sin\left(\frac{\sigma+\sigma_1}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma}{2}\right)}\tan\frac{\theta_1-\theta}{2}, \tag{104}$$

for all values of $\sigma$ except $\sigma_1$. For this exceptional case, the right-hand side is indeterminate, but Bianchi shows† that, if one substitute in place of $\tan\frac{\theta_1-\theta}{2}\Big/\sin\frac{\sigma_1-\sigma}{2}$, which assumes the form 0/0, the quotient of the derivatives with respect to $\sigma$, the formula becomes

$$\tan\frac{\phi-\omega}{2}=\sin\sigma_1\left[\frac{\partial\theta}{\partial\sigma}+c'\frac{\partial\theta}{\partial c}\right]_{\sigma=\sigma_1}, \tag{105}$$

where $c'$ is a new constant; and then it is shown that this function $\phi$ satisfies the condition of the problem. In considering the case where $\sigma=\sigma_1$, we mean a repetition of the same transformation, that is, the constant $c$ must have the same value also. Otherwise, the factor $\tan\left(\frac{\theta_1-\theta}{2}\right)$ doesn't vanish. For the latter case, we have $\phi=\omega+\pi$, which is an evident solution of the equations for $S_1$ similar to (88) for $S$.

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* Lezioni, p. 437; Ger. trans., p. 464. The difference between (104) and the formulae of Bianchi is due to the definition of the angle $\sigma$.

† L. c., p. 439; German trans., 467.

It is our purpose now to study the theorem of permutability with regard to its bearing upon the transformations of the $A$-surfaces. We consider first the exceptional case which leads to the formula (105).

Let $S$ be an $A$-surface with spherical representation (4) and denote by $\Sigma$ the pseudospherical surface with the same spherical representation. Corresponding to the solution $\theta_1=\theta(u, v, \sigma_1, c_1)$ of the system (88), there exists a transform $S_1'$, of the original surface, and a transform $\Sigma_1$ of $\Sigma$. The function $\phi$ given by (105) leads to a Bäcklund transformation of $\Sigma_1$ into a pseudospherical surface $\Sigma_2$. But we have seen that the function $\theta$, by means of which a Bäcklund transformation is given, serves to determine a transformation of all the $A$-surfaces in the same group with the given pseudospherical surface into $A$-surfaces with the same spherical representation as this Bäcklund transform. Hence $S_1'$ and all the $A$-surfaces with the same spherical representation are transformed by means of the function $\phi$ of (105) into surfaces of the same class as $\Sigma_2$.

We consider now the general case where $\phi$ is given by (103). As before, we denote by $S_1'$ the transform of $S$ by $\theta(u, v, \sigma_1, c_1)$, and by $S_1''$ the transform by means of $\theta(u, v, \sigma_2, c_2)$; for brevity, we write

$$\theta_1=\theta(u, v, \sigma_1, c_1), \quad \theta_2=\theta(u, v, \sigma_2, c_2).$$

Denote by $\Sigma_1'$ and $\Sigma_1''$ the pseudospherical surfaces into which $\Sigma$ is transformed by means of these respective functions. By the theorem of permutability, we know that if $\Sigma_1'$ is transformed by means of $\phi$ and angle $\sigma_2$, we get the same surface $\Sigma_2$ as when $\Sigma_1''$ is transformed by means of $\phi$ and $\sigma_1$. But when $S_1'$ is transformed by means of $\phi$ and $\sigma_2$ the transform $S_2'$ corresponds with $\Sigma_2$ with parallelism of tangent planes; similarly, the surface $S_2''$, which is the transform of $S_1''$ by $\phi$ and $\sigma_2$. Hence, $S_2'$ and $S_2''$ are $A$-surfaces of the same class. It remains to be discovered in what cases $S_2'$ and $S_2''$ coincide. We proceed as in the case of pseudospherical surfaces.

Denote by $\lambda'$ and $\mu'$ the values of $\lambda$ and $\mu$ when $S$ is transformed into $S_1'$; then, from (91),

$$\lambda' = (A\cos\omega + C\sin\omega)\sin\sigma_1, \quad \mu' = (-A\sin\omega + C\cos\omega)\sin\sigma_1. \tag{106}$$

In like manner, we denote by $\lambda_1'$, $\mu_1'$ the corresponding functions for $S_1'$, when the latter is transformed into $S_2''$; from (91) it follows that

$$\lambda_1' = (A_1'\cos\theta_1 + C_1'\sin\theta_1)\sin\sigma_2, \quad \mu_1' = (-A_1'\sin\theta_1 + C_1'\cos\theta_1)\sin\sigma_2, \tag{107}$$

where $A_1'$ and $C_1'$ are given by (93) when $\sigma$, $\theta$, $\lambda$, $\mu$ are given the values $\sigma_1$, $\theta_1$, $\lambda'$, $\mu'$.

In a manner similar to (80), we find for the projections, upon the tangents to the lines of curvature of $S_1'$, of the line $M_1'M_2'$ the values

$$\lambda_1'\cos\phi - \mu_1'\cos\sigma_2\sin\phi, \quad \lambda_1'\sin\phi + \mu_1'\cos\sigma_2\cos\phi, \quad \mu_1'\sin\sigma_2.$$

As in the case of pseudospherical surfaces, we find that the tangents to the lines of curvature of $S_1'$ make with the directions $MP$, $MQ$, $MN$ of $S$ angles whose cosines are given by (98). Consequently, the coordinates of $M_2'$ with respect to these directions for $S$ as axes are

$$\begin{aligned}
\xi &= \lambda' + \lambda_1'\cos(\phi-\omega) - \mu_1'\cos\sigma_2\sin(\phi-\omega),\\
\eta &= \mu'\cos\sigma_1 + \lambda_1'\cos\sigma_1\sin(\phi-\omega) + \mu_1'\cos\sigma_1\cos\sigma_2\cos(\phi-\omega) - \sin\sigma_1\sin\sigma_2\mu_1',\\
\zeta &= \mu'\sin\sigma_1 + \lambda_1'\sin\sigma_1\sin(\phi-\omega) + \mu_1'\sin\sigma_1\cos\sigma_2\cos(\phi-\omega) + \cos\sigma_1\sin\sigma_2\mu_1'.
\end{aligned}$$

And when we pass to the axes of the original trihedron associated with $S$ and denote by $x_2'$, $y_2'$, $z_2'$, the coordinates of $M_2'$ with respect to these axes, we find

$$\left.\begin{aligned}
x_2' &= \lambda'\cos\theta_1 - \mu'\cos\sigma_1\sin\theta_1 + (\lambda_1'\cos\theta_1 - \mu_1'\cos\sigma_1\cos\sigma_2\sin\theta_1)\cos(\phi-\omega)\\
&\qquad - (\mu_1'\cos\sigma_2\cos\theta_1 + \lambda_1'\cos\sigma_1\sin\theta_1)\sin(\phi-\omega) + \sin\sigma_1\sin\sigma_2\mu_1'\sin\theta_1,\\
y_2' &= \lambda'\sin\theta_1 + \mu'\cos\sigma_1\cos\theta_1 + (\lambda_1'\sin\theta_1 + \mu_1'\cos\sigma_1\cos\sigma_2\cos\theta_1)\cos(\phi-\omega)\\
&\qquad - (\mu_1'\cos\sigma_2\sin\theta_1 - \lambda_1'\cos\sigma_1\cos\theta_1)\sin(\phi-\omega) - \sin\sigma_1\sin\sigma_2\mu_1'\cos\theta_1,\\
z_2' &= \mu'\sin\sigma_1 + \lambda_1'\sin\sigma_1\sin(\phi-\omega) + \mu_1'\sin\sigma_1\cos\sigma_2\cos(\phi-\omega) + \cos\sigma_1\sin\sigma_2\mu_1'.
\end{aligned}\right\} \tag{108}$$

The coordinates of $S_2''$, which we shall denote by $x_2''$, $y_2''$, $z_2''$, can be found in the same manner. They can be obtained from (108) by putting double prime for the primes and interchanging the subscripts 1 and 2 of the $\theta$'s and $\sigma$'s. For $S_2'$ and $S_2''$ to be coincident, these corresponding coordinates must be equal, or what is the same thing, we must have

$$\begin{aligned} \cos\theta_1\,(x_2'-x_2'')+\sin\theta_1\,(y_2'-y_2'')&=0, \\ \cos\theta_2\,(x_2'-x_2'')+\sin\theta_2\,(y_2'-y_2'')&=0, \end{aligned} \quad z_2'=z_2''.$$

When the values for $x_2', \ldots, x_2'' \ldots, z_2''$ are put in this equation, they become

$$\left.\begin{aligned} &[\lambda_1''\cos(\theta_2-\theta_1)-\mu_1''\cos\sigma_1\cos\sigma_2\sin(\theta_2-\theta_1)-\lambda_1']\cos(\phi-\omega) \\ &\qquad+[\mu_1'\cos\sigma_2-\mu_1''\cos\sigma_1\cos(\theta_2-\theta_1) \\ &\qquad-\lambda_1''\cos\sigma_2\sin(\theta_2-\theta_1)]\sin(\phi-\omega) \\ &\quad=\lambda'-\lambda''\cos(\theta_2-\theta_1)+\mu''\cos\sigma_2\sin(\theta_2-\theta_1) \\ &\qquad-\sin\sigma_1\sin\sigma_2\mu_1''\sin(\theta_2-\theta_1), \\ &[\lambda_1'\cos(\theta_2-\theta_1)+\mu_1'\cos\sigma_1\cos\sigma_2\sin(\theta_2-\theta_1)-\lambda_1'']\cos(\phi-\omega) \\ &\qquad+[\mu_1''\cos\sigma_1-\mu_1'\cos\sigma_2\cos(\theta_2-\theta_1) \\ &\qquad+\lambda_1'\cos\sigma_1\sin(\theta_2-\theta_1)]\sin(\phi-\omega) \\ &\quad=\lambda''-\lambda'\cos(\theta_2-\theta_1)-\mu'\cos\sigma_1\sin(\theta_2-\theta_1) \\ &\qquad+\sin\sigma_1\sin\sigma_2\mu_1'\sin(\theta_2-\theta_1), \\ &[\mu_1'\sin\sigma_1\cos\sigma_2-\mu_1''\sin\sigma_2\cos\sigma_1]\cos(\phi-\omega) \\ &\qquad+[\lambda_1'\sin\sigma_1-\lambda_1''\sin\sigma_2]\sin(\phi-\omega) \\ &\quad=\sin\sigma_2(\mu''-\mu_1'\cos\sigma_1)-\sin\sigma_1(\mu'-\mu_1''\cos\sigma_2). \end{aligned}\right\} \quad (109)$$

When the $\lambda$'s, $\mu$'s, $\sin(\phi-\omega)$ and $\cos(\phi-\omega)$ are given the values determined by (106), (107) and (102), these equations take the forms

$$\begin{aligned} &L_1\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)+M_1\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+N_1(A\cos\omega+C\sin\omega) \\ &\qquad\qquad+R_1(-A\sin\omega+C\cos\omega)=0, \\ &L_2\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)+M_2\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+N_2(A\cos\omega+C\sin\omega) \\ &\qquad\qquad+R_2(-A\sin\omega+C\cos\omega)=0, \\ &L_3\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)+M_3\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)=0, \end{aligned}$$

where $L_1, \ldots, M_3$ are functions independent of $A$, $C$ and $\omega$. But when these various functions are calculated, it is found that they are all zero, so that in every case $S_2'$ and $S_2''$ coincide; we call it $S_2$. Hence the generalized theorem of permutability for $A$-surfaces.

*When the $A$-surface, which is the transform by means of $\theta_1$ of a given $A$-surface, $S$, is transformed by means of $\phi$, given by* (103), *and the angle $\sigma_2$, the new surface is the same as the transform by means of $\phi$ and angle $\sigma_1$ of the surface into which $S$ is transformed by means of $\theta_2$.*

We have remarked that, when $\omega$ is a solution of equation (7), $\omega + \pi$ also is a solution. As $\theta_1$ and $\theta_2$ in (103) satisfy the relation $\theta_1 + \theta_2 \neq (2m+1)\pi$, in order that $\phi$ may have the value $\omega + \pi$ it is necessary that

$$\sigma_2 = \sigma_1 + 2m\pi.$$

When these values are substituted in (96) and (100), both of the latter reduce to (88) in which $\theta$ is replaced by $\theta_1$ and $\theta_2$ respectively, and

$$\sigma = \sigma_1 = \sigma_2 - 2\pi m.$$

Thus all the equations of condition are satisfied for this value of $\phi$, and, furthermore, it is in no way dependent upon $\theta_1$ and $\theta_2$. We shall show that this is true also of the surface $S_2$ which evidently belongs to the same group of surfaces as the original surface $S$.

For the sake of brevity, we put

$$a = \sin^2\sigma_1\left(\frac{\partial A}{\partial u} + C\frac{\partial \omega}{\partial u}\right), \quad b = \sin^2\sigma_1\left(\frac{\partial C}{\partial v} - A\frac{\partial \omega}{\partial v}\right).$$

The function $\lambda_1$ and $\mu_1$ for the transformations of $S$ are

$$\lambda_1 = (A\cos\omega + C\sin\omega)\sin\sigma_1, \quad \mu_1 = (-A\sin\omega + C\cos\omega)\sin\sigma_1,$$

and the functions $\lambda_1'$, $\mu_1'$ for the transformation of $S_1$ are

$$\lambda_1' = a\cos\theta_1 + b\sin\theta_1 + \lambda_1,$$
$$\mu_1' = -a\sin\theta_1 + b\cos\theta_1 + \mu_1\cos\sigma_1.$$

When these values are substituted in (108), we get for the projections of $MM_2$ upon the axes of the trihedron of $S$ the values

$$-a, \quad -b, \quad (-A\sin\omega + C\cos\omega)\sin^2\sigma.$$

Hence if $x$, $y$, $z$ denote the coordinates of a surface $S$ with the linear element (1) and the spherical representation of its lines of curvature determined by $\omega$, the surface whose coordinates are of the form

$$x - \sin^2\sigma\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right)X_1 - \sin^2\sigma\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right)X_2 + (-A\sin\omega + C\cos\omega)\sin^2\sigma X \qquad (109')$$

is an $A$-surface with the same representation of its lines of curvature. Hence, when one has given an $A$-surface, one can find at once an infinity of $A$-surfaces with the same representation of their lines of curvature as those of the given surface; their coordinates are of the above form in which $\sigma$ is constant for each surface, but varies with the surface.

An exception to the foregoing arises in the case of pseudospherical surfaces, for when

$$A = \cos\omega, \quad C = \sin\omega,$$

the transform is the same as the original surface. Conversely, this is possible only in case

$$\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u} = 0, \quad \frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v} = 0, \quad -A\sin\omega + C\cos\omega = 0.$$

If we replace the last of these equations by

$$A = t\cos\omega, \quad C = t\sin\omega,$$

it is found from the first that $t$ is constant; hence, the surface must be pseudospherical.

If the surface is a parallel of a pseudospherical surface, then (23),

$$A = \cos\omega - a\sin\omega, \quad C = \sin\omega + a\cos\omega,$$

where $a$ is a constant. For this case the coordinates of the transforms are

$$x + a\sin^2\sigma X, \quad y + a\sin^2\sigma Y, \quad z + a\sin^2\sigma Z.$$

Hence the transforms are the surfaces parallel to the given surface, and when $\sigma = \pi/2$, the pseudospherical surface of the group is defined.

We consider, furthermore, the case where the given surface is a surface of Bianchi, defined by (56) or (64). In consequence of the relations (46) and (50), one finds for the coordinates of the transforms after the above manner

$$x\cos^2\sigma\,,\quad s\cos^2\sigma\,,\quad z\cos^2\sigma\,.$$

Hence, all the transforms are homothetic to the given surface, and when $\sigma=\pi/2$ the origin is the transform.

It was seen that the transforms of the parallels of pseudospherical surfaces are not homothetic to the original surface, hence it is of interest to seek the conditions to be satisfied in order that the transforms be homothetic to the given surface.

From (109′) it is seen that the necessary and sufficient condition for this is that the coordinates of the surface be of the form

$$\begin{aligned}
x&=\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)X_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)X_2+(A\sin\omega-C\cos\omega)\,X,\\
y&=\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)Y_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)Y_2+(A\sin\omega-C\cos\omega)\,Y,\\
z&=\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)Z_1+\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)Z_2+(A\sin\omega-C\cos\omega)\,Z,
\end{aligned}$$

when the linear element is

$$ds^2=A^2du^2+C^2dv^2.$$

The surface with the above coordinates is generated by the point whose coordinates are

$$\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\,,\quad\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\,,\quad A\sin\omega-C\cos\omega$$

with respect to the trihedron with fixed vertex and which rotates so as to have its axes parallel to the axes of the trihedron of any $A$-surface whose spherical representation is determined by the same $\omega$. The projections on these axes of a displacement are given by (40′), when $\lambda\cos\theta$, $\lambda\sin\theta$, $\mu$ are replaced by the above value. This gives

$$\begin{gathered}
\left[\frac{\partial}{\partial u}\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)-\frac{\partial\omega}{\partial v}\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+\sin\omega\,(A\sin\omega-C\cos\omega)\right]du+0\,.\,dv,\\
0\,.\,du+\left[\frac{\partial}{\partial v}\left(\frac{\partial C}{\partial v}-A\frac{\partial\omega}{\partial v}\right)+\frac{\partial\omega}{\partial u}\left(\frac{\partial A}{\partial u}+C\frac{\partial\omega}{\partial u}\right)-\cos\omega\,(A\sin\omega-C\cos\omega)\right]dv,\\
0\,du+0\,dv\,.
\end{gathered}$$

From the conditions of our problem the coefficient $du$ in the first must be $A$ and of $dv$ in the second $C$. Put

$$\phi = A \cos \omega + C \sin \omega ,$$

then the equations of condition become

$$\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial \log \cos \omega}{\partial u} \frac{\partial \phi}{\partial u} - \frac{\partial \log \sin \omega}{\partial v} \frac{\partial \phi}{\partial v} - \cos^2 \omega . \phi = 0 ,$$
$$\frac{\partial^2 \phi}{\partial v^2} - \frac{\partial \log \cos \omega}{\partial u} \frac{\partial \phi}{\partial u} - \frac{\partial \log \sin \omega}{\partial v} \frac{\partial \phi}{\partial v} - \sin^2 \omega . \phi = 0 .$$

In consequence of the relation

$$A \frac{\partial \omega}{\partial u} = \frac{\partial C}{\partial u} ,$$

when a common solution of the above equations is known, the function $C$ is given by the linear equation

$$\frac{\partial C}{\partial u} \cos \omega + \sin \omega \frac{\partial \omega}{\partial u} , \quad C = \phi \frac{\partial \omega}{\partial u} ,$$

so that its determination requires two quadratures and then $A$ is found directly.

## §9.—*General Transformations of Surfaces of Bianchi.*

It was seen that the surfaces of Bianchi of the three types had for transforms of angle $\pi/2$ surfaces of the parabolic type. We consider now the transforms for any value of $\sigma$. And we begin by remarking that, when $\Sigma$ is of the parabolic type and the origin is the point through which all of the associated circles pass, the coefficients of the linear element may be given the form (58),

$$A = -(e^{-\alpha_1} \cos \omega - \gamma_1 \sin \omega) , \quad C = -(e^{-\alpha_1} \sin \omega + \gamma_1 \cos \omega) , \tag{58}$$

where the expressions for $\alpha_1$ and $\gamma_1$ are given by (46) and (50). If we effect upon $\Sigma$ a transformation of angle $\sigma$ and by means of $\theta$, given by (88), the values of $\lambda$ and $\mu$ are

$$\lambda = - e^{-\alpha_1} \sin \sigma , \quad \mu = - \gamma_1 \sin \sigma ,$$

and the coefficients of the linear element of the transform $\Sigma_1$ are found from (93) to have the form

$$\left.\begin{aligned} A_1 &= \sin\sigma\, e^{-\alpha_1}\cos\theta_1 - e^{-\alpha_1}\cos\theta + \gamma_1\cos\sigma\sin\theta, \\ C_1 &= \sin\sigma\, e^{-\alpha_1}\sin\theta_1 - e^{-\alpha_1}\sin\theta - \gamma_1\cos\sigma\cos\theta. \end{aligned}\right\} \tag{110}$$

From the general defining theorem of Bianchi, it follows that if one of the transforms of $\Sigma_1$, for $\sigma$ a right angle, reduces to a point, then $\Sigma_1$ is a surface of Bianchi of the parabolic type.

For such a transformation the value of $\lambda_1$ is

$$\lambda_1 = \sin\sigma\, e^{-\alpha_1}\cos(\theta_1 - \theta) - e^{-\alpha_1},$$

and the coefficients of the linear element of transform are found by (44) to be

$$\begin{aligned} A_2 &= e^{-\alpha_1}\{\cos\omega\cos(\theta_1-\theta) - \cos\sigma\sin\omega\sin(\theta_1-\theta) - \sin\sigma\cos\omega + \lambda_1 e^{\alpha_1}\cos\phi\}, \\ C_2 &= e^{-\alpha_1}\{\sin\omega\cos(\theta_1-\theta) + \cos\sigma\cos\omega\sin(\theta_1-\theta) - \sin\sigma\sin\omega + \lambda_1 e^{\alpha_1}\sin\phi\}, \end{aligned}$$

where $\phi$ is a solution of system (48). For one of these transforms to reduce to a point there must be a solution $\phi$ of (48) which makes both of the above expressions zero. Multiply the first by $\cos\omega$, the second by $\sin\omega$ and add; similarly multiply by $-\sin\omega$, $\cos\omega$ respectively, and add; this gives

$$\left.\begin{aligned} [\sin\sigma\cos(\theta_1-\theta) - 1]\cos(\phi-\omega) &= \sin\sigma - \cos(\theta_1-\theta), \\ [\sin\sigma\cos(\theta_1-\theta) - 1]\sin(\phi-\omega) &= -\cos\sigma\sin(\theta_1-\theta). \end{aligned}\right\} \tag{111}$$

When these are compared with (102), it is found that the latter take the above form when $\sigma_2$ is $\pi/2$ and $\theta_1$ is the same as $\theta_2$ in (102). Hence:

*When a surface of Bianchi of the parabolic type is transformed under any angle $\sigma$, the transform is a surface of the same kind.* And if $\theta$ and $\sigma$ are the functions defining the transformation, and $\theta_1$ is the function appearing in the expression for the coefficients of the linear element (58), the function $\phi$ which leads to the point surface of the transforms of the new surface is given by (103) in the form

$$\tan\left(\frac{\phi-\omega}{2}\right) = \frac{\sin\left(\frac{\pi}{4}+\frac{\sigma}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\sigma}{2}\right)}\tan\left(\frac{\theta_1-\theta}{2}\right).$$

Let $\Sigma$ be a surface of Bianchi of any type whatever. Denote by $S_1$ its transform of angle $\sigma$ and function $\theta_1$ and by $\Sigma_1$ its transform by $\theta'$ where the latter is a solution of equations (38). From preceding results we know that $\Sigma_1$ is a surface of Bianchi of the parabolic type. The foregoing theorem tells us that, if we transform $\Sigma_1$ under angle $\sigma$ and by means of $\phi$ given by (102) which now take the form

$$\left.\begin{aligned} \cos(\phi-\omega) &= \frac{\sin\sigma-\cos(\theta'-\theta)}{\sin\sigma\cos(\theta'-\theta)-1}, \\ \sin(\phi-\omega) &= -\frac{\cos\sigma\sin(\theta'-\theta)}{\sin\sigma\cos(\theta'-\theta)-1}, \end{aligned}\right\} \tag{112}$$

the resulting surface $\Sigma_2$ will be a surface of Bianchi of the parabolic type. But this surface $\Sigma_2$ is also the transform of $S_1$ for $\sigma = \pi/2$ and same $\phi$. By varying $\theta$ and keeping $\theta$ fixed we get an infinty of surfaces $\Sigma_2$ of the parabolic type which are orthogonal to the circles associated with $S_1$. From the theorem of Bianchi it follows that one of the surfaces $\Sigma_2$ must reduce to a point in order that $S_1$ be a surface of Bianchi of the parabolic type. We procced to the determination of the cases where this is possible.

We begin with the general definition (64) of surfaces of Bianchi, whose linear element has for coefficients the squares of the quantities (66),

$$\left.\begin{aligned} A &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\cos\omega + t\sin\omega, \\ C &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\omega - t\cos\omega. \end{aligned}\right\} \tag{66}$$

The functions $\lambda$ and $\mu$, which figure in the transformation of angle $\sigma$ of these surfaces, have the forms

$$\lambda = \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\sigma, \qquad \mu = -t\sin\sigma,$$

and the quantities $A_1$ and $C_1$ for the transform $S_1$ are found by (93) to be

$$\left.\begin{aligned} A_1 = \sin\sigma\,&[\tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\cos\theta_1+\beta\sin\theta_1] \\ &\qquad + \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\cos\theta + t\cos\sigma\sin\theta, \\ C_1 = \sin\sigma\,&[\tfrac{1}{2}\{e^{\alpha}-(\beta^2+k)e^{-\alpha}\}\sin\theta_1-\beta\cos\theta_1] \\ &\qquad + \tfrac{1}{2}\{e^{\alpha}+(\beta^2+k)e^{-\alpha}\}\sin\theta - t\cos\sigma\cos\theta. \end{aligned}\right\} \tag{113}$$

In order to get the surfaces orthogonal to the circles associated with $S_1$, we have only to transform $S_1$ under angle $\sigma = \pi/2$ and by $\phi$, given by (112). For this transformation $\lambda_1$ has the value

$$\lambda_1 = \sin\sigma\left[\tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\right\}\cos(\theta_1 - \theta) + \beta\sin(\theta_1 - \theta)\right] + \tfrac{1}{2}\left\{e^{\alpha} + (\beta^2 + k)\,e^{-\alpha}\right\}. \tag{114}$$

In calculating the functions $A_2$ and $C_2$ for the new surface, by means of formulæ similar to (44), it must be remembered that $\theta_1$ satisfies equations (38) and $\theta$ equations (88). These functions are found to be of the form

$$\begin{aligned} A_2 = {} & \left[\sin\sigma\,\tfrac{1}{2}\left\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\right\} + \tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\right\}\cos(\theta_1 - \theta)\right. \\ & \left. + \beta\sin(\theta_1 - \theta)\right]\cos\omega - \cos\sigma\left[\tfrac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\sin(\theta_1 - \theta)\right. \\ & \left. - \beta\cos(\theta_1 - \theta)\right]\sin\omega + \lambda_1\cos\phi, \\ C_2 = {} & \left[\sin\sigma\,\tfrac{1}{2}\left\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\right\} + \tfrac{1}{2}\left\{e^{\alpha} - (\beta^2 + k)\,e^{-\alpha}\right\}\cos(\theta_1 - \theta)\right. \\ & \left. + \beta\sin(\theta_1 - \theta)\right]\sin\omega + \cos\sigma\left[\tfrac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\sin(\theta_1 - \theta)\right. \\ & \left. - \beta\cos(\theta_1 - \theta)\right]\cos\omega + \lambda_1\sin\phi. \end{aligned}$$

For one of these surfaces to reduce to a point, it must be possible to find a function $\phi$ which makes $A_2$ and $C_2$ zero. Equate the above expressions to zero; multiply by $\cos\omega$ and $\sin\omega$ respectively, and add; again multiply by $-\sin\omega$, $\cos\omega$ and add; this gives

$$\left.\begin{aligned} & \lambda_1\cos(\phi - \omega) + \sin\sigma\,\tfrac{1}{2}\left\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\right\} \\ & \qquad + \tfrac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\cos(\theta_1 - \theta) + \beta\sin(\theta_1 - \theta) = 0, \\ & \lambda_1\sin(\phi - \omega) + \cos\sigma\left[\tfrac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\sin(\theta_1 - \theta)\right. \\ & \qquad \left. - \beta\cos(\theta_1 - \theta)\right] = 0. \end{aligned}\right\} \tag{115}$$

Expressing the condition that

$$\sin^2(\phi - \omega) + \cos^2(\phi - \omega) = 1,$$

it is found that we must have $k$ equal to zero. Hence the theorem:

*The surfaces of Bianchi of any type are transformed into surfaces of the parabolic type when $\sigma$ is $\pi/2$, and only surfaces of the parabolic type have such transforms for any value of $\sigma$.*

It only remains to point out the transformation function $\theta'$ which gives $\phi$ such a form that $\Sigma_2$ reduces to a point. If we substitute in (115) for $\cos(\phi - \omega)$

and sin ($\phi - \omega$) the values (112), the former can be reduced to

$$\left.\begin{aligned} &\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)\, e^{-\alpha}\} \cos(\theta' - \theta) + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \cos(\theta_1 - \theta) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \beta \sin(\theta_1 - \theta) = 0\,, \\ &\sin\sigma\, [\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \sin(\theta_1 - \theta') - \beta \cos(\theta_1 - \theta')] \\ &\qquad - \tfrac{1}{2}\{e^{\alpha} - e^{-\alpha}\, (\beta^2 + k)\} \sin(\theta_1 - \theta) \\ &\qquad - \tfrac{1}{2}\{e^{\alpha} + e^{-\alpha}\, (\beta^2 + k)\} \sin(\theta' - \theta) + \beta \cos(\theta_1 - \theta) = 0\,. \end{aligned}\right\} \qquad (116)$$

Since this must be true for all values of $\sigma$, the last equation reduces to the two

$$\left.\begin{aligned} &\tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \sin(\theta_1 - \theta') - \beta \cos(\theta_1 - \theta') = 0, \\ &\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)\, e^{-\alpha}\} \sin(\theta' - \theta) + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \sin(\theta_1 - \theta) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - \beta \cos(\theta_1 - \theta) = 0\,. \end{aligned}\right\} \qquad (117)$$

If the first of (116) be multiplied by cos $\theta$ and the last of (117) by sin $\theta$ and the resulting equations subtracted, and the former of these be multiplied by sin $\theta$ and the latter by cos $\theta$ and the results added we get

$$\left.\begin{aligned} &\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)\, e^{-\alpha}\} \cos\theta' + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \cos\theta_1 + \beta \sin\theta_1 = 0\,, \\ &\tfrac{1}{2}\{e^{\alpha} + (\beta^2 + k)\, e^{-\alpha}\} \sin\theta' + \tfrac{1}{2}\{e^{\alpha} - (\beta^2 + k)\, e^{-\alpha}\} \sin\theta_1 - \beta \cos\theta_1 = 0\,. \end{aligned}\right\} \qquad (118)$$

When these values are substituted in the first of (117), it is satisfied.

A comparison of these results with the discussion of the transformations of surfaces of Bianchi, developed in §6, will reveal the fact that the function $\theta'$, which gives $\phi$ such a form that $\Sigma_2$ shall be a point, is the very function which transforms $\Sigma$ of the parabolic type into a point. A similar result obtains in the discussion of (111).

## §10.—*Transformations of a Particular Class of Surfaces.*

The surfaces to be considered in this section are suggested by a similar class of pseudospherical surfaces studied by Darboux* and Bianchi.† We start with the remark that an evident solution of equation (7) is $\omega = 0$. From the general expression for the linear element of pseudospherical surfaces

$$ds^2 = \cos^2 \omega du^2 + \sin^2 \omega dv^2,$$

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* Leçons, Vol. 3, p. 463.

† Lezioni, p. 440 ; Ger. trans., 466.

it follows that the corresponding surface of this kind for the above solution is a curve of parameter $u$. We take the very special case where this curve is a straight line, and let it be the $z$-axis, then the rectangular coordinates of the surface are

$$x = 0, \quad y = 0, \quad z = u.$$

The tangent planes to this degenerate surface pass through the $z$-axis, hence the tangent planes to the other $A$-surfaces belonging to the solution $\omega = 0$ are cylinders whose generators are parallel to this line. Consequently, the most general surfaces of this group are given by

$$x = V \cos v, \quad y = V \sin v, \quad z = u,$$

where $V$ is a function of $v$ alone whose form will be determined later. The linear element of the surface has the form

$$ds^2 = du^2 + (V'^2 + V^2)\, dv^2,$$

where the accent denotes differentiation. From the above expressions we get for the direction-cosines of the normal to the surface

$$X = -\frac{V' \sin v + V \cos v}{\sqrt{V^2 + V'^2}}, \quad Y = \frac{V' \cos v - V \sin v}{\sqrt{V^2 + V'^2}}, \quad Z = 0, \tag{119}$$

and denoting by $X_1, Y_1, Z_1$; $X_2, Y_2, Z_2$ the direction-cosines of the tangents to the lines of curvature $v =$ const., $u =$ const. respectively, we have

$$\left.\begin{aligned} &X_1 = 0, \quad Y_1 = 0, \quad Z_1 = 1, \\ &X_2 = \frac{V' \cos v - V \sin v}{\sqrt{V^2 + V'^2}}, \quad Y_2 = \frac{V' \sin v + V \cos v}{\sqrt{V^2 + V'^2}}, \quad Z_2 = 0. \end{aligned}\right\} \tag{120}$$

From (119) we have

$$\left.\begin{aligned} \frac{\partial X}{\partial v} &= \frac{(V^2 + 2V'^2 - VV'')(V \sin v - V' \cos v)}{(V^2 + V'^2)^{\frac{3}{2}}}, \\ \frac{\partial Y}{\partial v} &= -\frac{(V^2 + 2V'^2 - VV'')(V \cos v + V' \sin v)}{(V^2 + V'^2)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{121}$$

But from (4), it is seen that for this case we must have

$$\Sigma \left(\frac{\partial X}{\partial v}\right)^2 = 1,$$

so that $V$ must satisfy the condition

$$[V^2 + 2V'^2 - VV'']^2 = (V^2 + V'^2)^2.$$

A comparison of (120), (121) and (55) shows that it may be replaced by

$$V^2 + 2V'^2 - VV'' = V^2 + V'^2,$$

from which we have

$$V = ae^{cv}.$$

Hence the $A$-surfaces of the same class as the $z$-axis are given by

$$x = ae^{cv}\cos v, \quad y = ae^{cv}\sin v, \quad z = u, \tag{122}$$

where $a$ and $c$ are arbitrary constants, and the linear element is

$$ds^2 = du^2 + a^2e^{2cv}(1 + c^2)\,dv^2. \tag{123}$$

For the present case, the equations (88) reduce to

$$\frac{\partial\theta}{\partial u} = \frac{\sin\theta}{\sin\sigma}, \quad \frac{\partial\theta}{\partial v} = \frac{\cos\sigma}{\sin\sigma}\sin\theta,$$

of which the general integral is

$$\tan\frac{\theta}{2} = Ce^{\dfrac{u + \cos\sigma\,.\,v}{\sin\sigma}},$$

where $C$ is an arbitrary constant. When $C$ is zero, $\theta$ is zero or $\pi$, and hence gives the surface from which we start, we take $C$ equal to unity. From (123) we have

$$\lambda\sin\sigma, \quad \mu = a\sqrt{1 + c^2}\,e^{cv}\sin\sigma,$$

so that, by means of (80), (119) and (120), we find for the coordinates of the transforms of these surfaces

$$\left.\begin{aligned}
x_1 &= ae^{cv}\cos v + \frac{c\cos v - \sin v}{\sqrt{1+c^2}}\,\frac{\sin v}{\cosh\alpha} \\
&\qquad - ae^{cv}\sin\sigma\,[(c\cos v - \sin v)\cos\sigma\tanh\alpha \\
&\qquad + \sin\sigma\,(c\sin v + \cos v)], \\
y_1 &= ae^{cv}\sin v + ae^{cv}\sin\sigma\,[(c\cos v - \sin v)\sin\sigma \\
&\qquad - \cos\sigma\tanh\alpha\,(c\sin v + \cos v)] + \frac{c\sin v + \cos v}{\sqrt{1+c^2}}\,\frac{\sin\sigma}{\cosh\alpha}, \\
z_1 &= u - \sin\sigma\tanh\alpha - \frac{a\sqrt{1+c^2}\,e^{cv}\sin\sigma\cos\sigma}{\cosh\alpha},
\end{aligned}\right\} \tag{124}$$

where we have put, for the sake of brevity,

$$\alpha = \frac{u + \cos\sigma v}{\sin\sigma}.$$

When in (121) we put $c$ equal to zero, these formulæ define circular cylinders, so that (124) define all the transforms of circular cylinders. Furthermore, when we put $a$ also equal to zero, we have the original case, the $z$-axis; from (124) we have for the formulæ of its transforms

$$x_1 = -\frac{\sin v \sin\sigma}{\cosh\alpha}, \quad y_1 = \frac{\cos v \,.\, \sin\sigma}{\cosh\alpha}, \quad z_1 = u - \sin\sigma \tanh\alpha. \qquad (125)$$

Bianchi has shown* that the surfaces so defined are pseudospherical helicoidal surfaces whose meridian profile in the tractrix.

If, in particular, we put $\sigma$ equal to $\pi/2$ (125), define the pseudosphere.† In this case $\alpha$ and, therefore, $\theta$ is a function of $u$ alone, so that the surfaces defined by

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{1+c^2}\cosh u}(c\cos v - \sin v) - ace^{cv}\sin v, \\ y_1 &= \frac{1}{\sqrt{1+c^2}\cosh u}(c\sin v + \cos v) + ace^{cv}\cos v, \\ z_1 &= u - \tanh u, \end{aligned}$$

are *moulure* surfaces.

The complete determination of all the $A$-surfaces with the same spherical representation of their lines of curvature, as the surfaces defined by (124) requires the integration of the equation of the form (8), which, in this case, reduces to

$$\frac{\partial^2\psi}{\partial u\,\partial v} + \frac{e^{2\alpha}-1}{1+e^{2\alpha}}\,\frac{\cos\sigma}{\sin\sigma}\,\frac{\partial\psi}{\partial u} + \frac{4e^{2\alpha}}{1-e^{4\alpha}}\,\frac{\partial\psi}{\partial v} = 0.$$

It is readily found that the invariant $h$‡ for this equation is zero, so that the complete integral can be found at once; it is

$$\psi\,(e^{\alpha} - e^{-\alpha}) = \int (e^{\alpha} - e^{-\alpha})\,V dv + U,$$

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* Lezioni, p. 441; Ger. trans., p. 467.

† Bianchi, l. c., p. 443; Darboux, Leçons, Vol. 3, p. 465.

‡ Darboux. Leçons, Vol. 2, 23.

when $U$ and $V$ are arbitrary functions of $u$ and $v$ respectively. When this quadrature has been effected for each form of $V$, the further determination of these surfaces is direct and without integration. In consequence of the theorem of permutability, the transforms of all these surfaces can be found by algebraic processes and differentiation. Thus, if we effect upon the original surfaces (122) the transformation given by

$$\tan\frac{\theta_1}{2} = e^{\alpha_1}, \quad \alpha_1 = \frac{u + \cos\sigma_1 v}{\sin\sigma_1},$$

the functions $\phi$, which enter into the transformations of the resulting surfaces are given by (103), which now take the form

$$\tan\frac{\phi}{2} = \frac{\sin\left(\frac{\sigma_1+\sigma}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma}{2}\right)} \frac{e^{\alpha_1} - e^{\alpha}}{1 + e^{\alpha_1+\alpha}},$$

and for the particular case, $\sigma = \sigma_1$, we have by (105),

$$\tan\frac{\phi}{2} = \frac{c' - u\cos\sigma_1 - v}{\sin\sigma_1 \cos\alpha_1}.$$

The further transformations of the surfaces obtained by the application of the foregoing formulæ are merely a repetition of the preceding, so that all of the surfaces arising from (122) by transformations can be found directly.

It has been seen that certain moulure surfaces belong to the suite of surfaces which are given by the preceding transformations. In seeking for the transforms of these, we propose the more general question as to whether moulure surfaces can be transformed into surfaces of the same kind. To this end we take $\omega$, a function of $u$ alone. From the form, to which the first of equations (88) reduces, it is evident that for $S_1$ to be a moulure surface $\theta$ must be a function of $u$ alone. Then these equations reduce to

$$\left.\begin{aligned} \sin\sigma\frac{\partial\theta}{\partial u} &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega, \\ \sin\sigma\frac{\partial\omega}{\partial u} &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega. \end{aligned}\right\} \qquad (126)$$

From (17) it follows that $\omega$ and $\theta$ must satisfy also the equations

$$\frac{d\theta}{du} = \sqrt{\sin^2\theta - a^2}, \quad \frac{d\omega}{du} = \sqrt{\sin^2\omega - a_1^2}, \tag{127}$$

where $a$ and $a_1$ are constants. If equations (126) be squared and the values for $\frac{d\theta}{du}$ and $\frac{d\omega}{du}$ from (127) be substituted, we get two equations from which, by subtraction, we find that $a_1^2$ is equal to $a^2$. The first of these equations is

$$\sin^2\sigma\,(\sin^2\theta - a^2) = (\sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega)^2. \tag{128}$$

If this equation be differentiated with respect to $u$ and the values of $\frac{d\theta}{du}$ and $\frac{d\omega}{du}$ given by (126) be substituted, the equation is an identity. When $\sigma = \pi/2$, the above equation reduces to

$$\sin^2\theta\sin^2\omega = a^2.$$

Hence:

*A moulure surface can always be transformed into a moulure surface and the transformation can be effected by algebraic processes.*

It is clear to be sure that other $A$-surfaces can be transformed into moulure surfaces, just as all the transforms of a moulure surface are not surfaces of this kind.

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